PERSAMAAN DIFERENSIAL

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Editor:
Herna Febrianty Sianipar, M.Si

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RC
Rumah Cemerlang

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KATA PENGANTAR

Mengingat akan kesulitan - kesulitan yang sering dialami para mahasiswa eksakta dalam mempelajari persamaan diferensial, serta kurangnya buku- buku mengenai persamaan diferensial maka penulis berusaha mengatasinya dengan menyusun persamaan diferensial ini.

Dalam buku persamaan diferensial ini, penulis telah berusaha menyajikan bahan- bahan dalam bentuk pemecahan atau uraian- uraian yang mengandung unsur – unsur didaktik dan metodik yang singkat, praktis dan konstruktif dalam uraian- uraian pembuktian rumus maupun dalam penyelesaian soal.

Jadi semua uraian- uraiannya mudah dipelajari dan dimengerti, demikian pula contoh- contoh soalnya. Inilah yang membedakan buku ini denga buku-buku lain yang mengandung persamaan diferensial. Semoga dengan dibuatkan buku ini pembaca dapat memahami tentang definis, teorema, dan pernyataan secara detail dengan memperhatikan contoh-contoh soal yang mengiringinya. Cobalah untuk menjawab soal-soal yang ada pada setiap akhir bab. Silahkan belajar kelompok sebagai sarana diskusi sekaligus untuk menghilangkan rasa jenuh.

Akhirnya, diharapkan semoga buku ini bermanfaat bagi pembaca khususnya mahasiswa. Penulis juga mengharapkan kritik dan saran yang bersifat membangun demi pembuatan buku selanjutnya yang masih berhubungan dengan persamaan diferensial.

Pematang Siantar,   September 2021

Yanty M. R. Marbun, M.Pd

DAFTAR ISI

BAB I

PERSAMAAN DIFERENSIAL TINGKAT SATU DERAJAT SATU

Variabel Yang Dapat Dipisahkan

Bentuk Umum:

P (x,y)dx + Q (x,y)dy = 0

Persamaan ini dapat diselesaikan berdasarkan bentuknya, dimana terdiri dari 2 bentuk yaitu:

1. M (x)dx + N (y)dy = 0
   Fungsi yang mengandung dx hanya fungsi x, dan fungsi yang mengandung unsur dy hanya fungsi y, sehingga kedua ruas bisa langsung diintegralkan menjadi:
   $\int M(x)\, dx + \int N(y) dy = C$
2. $M(x).R(x)dx + N(x).S(y)dy = 0$
   Fungsi yang mengandung dx ada fungsi x, dan ada fungsi y, dan fungsi yang mengandung unsur dy ada fungsi x dan ada fungsi y, sehingga kedua ruas harus diubah dulu menjadi bentuk pertama. Dengan membagi kedua ruas dengan $R(y).N(x)$ menjadi:
   $\frac{M(x)}{N(x)} dx + \frac{S(y)}{R(y)} dy = 0 \longrightarrow \int \frac{M(x)}{N(x)} dx + \int \frac{S(y)}{R(y)} dy = C$

Contoh Soal:

1. $(xy - x)dx + (xy + y)dy = 0$

   Penyelesaian :

   Jawab: $\xrightarrow{x(y-1)dx+y(x+1)dy} : (y-1)(x+1)$

$$\int \frac{x}{x+1} dx + \int \frac{y}{y-1} dy = C_1$$

$$\int \frac{x+1-1}{x+1} dx + \int \frac{y-1+1}{y-1} dy = C_1$$

$$\int \frac{x+1}{x+1} dx - \int \frac{1}{x+1} dx + \int \frac{y-1}{y-1} dy + \int \frac{1}{y-1} dy = C_1$$

$$\int dx - \ln(x+1) + \int dy + \ln(y-1) = C_1$$
$$x - \ln(x+1) + y + \ln(y-1) = C_1$$
⇒misalkan C_1=ln C
$$x + y + \ln\frac{y-1}{x+1} = \ln C$$
$$e^{x+y+\ln\frac{y-1}{x+1}} = e^{\ln C}$$
$$e^x . e^y . e^{\ln\frac{y-1}{x+1}} = C$$
$$e^x . e^y . \frac{y-1}{x+1} = C$$
$$(y-1)e^x . e^y = C(x+1)$$

2. $y^3 \mathrm{dy} + \sqrt{1-x^2}\, \mathrm{dy} = 0$

Penyelesaian :
$y^3 \mathrm{dy} + \sqrt{1-x^2}\, \mathrm{dy} = 0$…………… dibagi $y^3(\sqrt{1-x^2}\ )$
$$\frac{1}{\sqrt{1-x^2}} \mathrm{dx} + \frac{1}{y^3} \mathrm{dy} = 0$$
$$\int \frac{1}{\sqrt{1-x^2}} \mathrm{dx} + \int \frac{1}{y^3} \mathrm{dy} = 0$$
$$\arcsin x + (-\frac{1}{2} y^{-2}) = C_1$$
$$\arcsin x - \frac{1}{2y^2} = C_1$$
$$\arcsin x - C_1 = \frac{1}{2y^2} \ldots\ldots\ldots\ldots\ldots.. \text{di kali } 2$$
$$2(\arcsin x - C_1) = \frac{1}{y^2}$$
$$2(\arcsin x - C_1)y^2 = 1 \quad , \text{misalkan} : -C_1 = C$$
$$y^2 = \frac{1}{2(\arcsin x + C)}$$

3. $y(1-x)dx + x^2(1-y)dy = 0$

Penyelesaian:
$$y(1-x)dx + x^2(1-y)dy = 0 \quad \longrightarrow dibgi\ dengan\ (y.x^2)$$
$$\int \frac{1-x}{x^2} dx + \int \frac{1-y}{y} dy = C_1$$
$$\int \frac{1}{x^2} dx - \int \frac{x}{x^2} dx + \int \frac{1}{y} dy - \int \frac{y}{y} dy = C_1$$
$$\int \frac{1}{x^2} dx - \int \frac{1}{x} dx + \int \frac{1}{y} dy - \int dy = C_1$$

$$-\frac{1}{x} - \ln x + \ln y - y = C_1$$

$$-\frac{1}{x} + \ln\frac{y}{x} - y = C_1$$ ⇒misalkan C_1=ln C

$$\ln\frac{y}{x} - y - \frac{1}{x} = \ln C$$

$$\ln\frac{y}{x} - \ln C = y + \frac{1}{x}$$

$$\ln\frac{y}{x.C} = y + \frac{1}{x}$$

Dimana,

$$e^{\ln} = \ln e = 1$$

$$e^{\ln\frac{y}{x.C}} = e^{y+\frac{1}{x}}$$

$$\frac{y}{x.C} = e^{y}.e^{\frac{1}{x}}$$

$$y = xCe^{y}.e^{\frac{1}{x}}$$

Latihan Soal:

Selesaikan solusi dari persamaan diferensial dibawah ini:

1. $(1+2x^2)y\,y' = 2x(1+y^2)$
2. $x^2y\,dx + (x + 1)dy = 0$
3. $y' + (y + 1)\cos x = 0$
4. $\sin x \cos y\, dx + \tan y \cos x\, dy = 0$
5. $\frac{dy}{dx} = \frac{4x+xy^2}{y- x^2y}$
6. $(x^2 + 4)\frac{dy}{dx} = (y + 2)\left(x + \sqrt{x^2 + 4}\right)$
7. $\frac{dy}{dx} = \frac{(2-y)^2}{2\sqrt{1+x}}$
8. $2xy(4 - y^2)dx + (y - 1)(x^2 + 2)dy = 0$

Persamaan Diferensial Homogen

Bentuk Umum :

$$M(x, y)dx + N(x, y)dy = 0$$

Ciri umum Homogen : Tiap suku derajatnya sama.

Contoh :

1. $F(x, y) \quad = 3x^2 + 4xy - 7y^2$

   $F(tx, ty) = 3t^2x^2 + 4t^2xy - 7t^2y^2$

   $F(tx, ty) = t^2(3x^2 + 4xy - 7y^2)$

   $t^2.F(x, y)$ ⟶ *Homogen derajat 2*

2. $F(x, y) \quad = x + \sqrt{x^2 + y^2}$

   $F(tx, ty) = tx + \sqrt{t^2x^2 + t^2y^2}$

$$= tx + t\sqrt{x^2+y^2} = t\left(x + \sqrt{x^2+y^2}\right)$$

$t.F(x,y)$ ⟶ *Homogen derajat 1*

3. $F(x,y) \ = x^2 + y$

   $F(tx,ty) = t^2x^2 + ty = t(tx^2 + y)$ ⟶ *Non Homogen*

Bentuk umum diatas dikatakan PD Homogen jika :

Fungsi M dan N adalah homogen dengan derajat sama. Persamaan ini dapat diselesaikan dengan subsitusi :

$v = \frac{y}{x}$ *atau* $y = vx$

$dy = vdx + xdv$

$M(x,y) \to M(x,vx) = x^m R(v)$

$N(x,y) \to N(x,vx) = x^m S(v)$

$x^m R(v)dx + x^m S(v)(vdx + xdv) = 0$

⟶ $: dibagi\ dengan\ x^m$

$R(v)dx + S(v)(vdx + xdv) = 0$

$R(v)dx + v.S(v)dx + x.S(v)dv = 0$

$\{R(v) + v.S(v)\}dx + x.S(v)dv = 0$

⟶ $: dibagi\ dengan\ \ x\ R(v) + v.S(v)$

$$\frac{dx}{x} + \frac{S(v)}{R(v) + vS(v)}dv = 0$$

Maka : $\int \frac{dx}{x} + \int \frac{SV}{R(V)+V.S(V)}dv = C$

Contoh soal :

Carilah jawab persamaan differensial :

1. $\frac{dy}{dx} = \frac{-xy+y^2}{xy}$

Penyelesaian:

$\frac{dy}{dx} = \frac{-xy+y^2}{xy}$

$xydy = (-xy + y^2)dx$

$xydy + (xy - y^2)dx = 0$ $\quad :dibagi\ dengan\ x^2$

$\frac{y}{x}dy + \frac{y}{x}dx - \left(\frac{y}{x}\right)^2 dx = 0$ ....(1)

Substitusi $v = \frac{y}{x} \implies y = vx$

$dy = vdx + xdv$

Persamaan (1) menjadi:

$v\ (vdx + xdv) + vdx - v^2dx = 0$

$v^2dx + vx\ dv + v\ dx - v^2dx = 0$

$vx\,dv + v\,dx = 0 \xrightarrow{\hspace{3cm}}$ dibagi $dengan\ vx$

$dv + \frac{dx}{x} = 0$

$\int dv + \int \frac{dx}{x} = C_1$

$v + lnx = C_1$, subsitusi nilai v = y/x

$\frac{y}{x} + x = C_1$, di ln-kan

$e^{\frac{y}{x}} + e^{lnx} = e^{C_1}$, misalkan $e^{C_1} = C$

$xe^{\frac{y}{x}} = C$

2. $\frac{dy}{dx} = \frac{y}{x} - cot\frac{y}{x}$

Penyelesaian :

$\left(\frac{y}{x} - cot\frac{y}{x}\right)dx = dy$

$\left(\frac{y}{x} - cot\frac{y}{x}\right)dx - dy = 0 \ldots\ldots\ldots\ldots.(1);$

$subsitusikan:\ v = \frac{y}{x}, y = vx, dy = vdx + xdv\ ke\ (1)$

$didapat:\ (v - cot\,v)dx - (vdx + xdv) = 0$

$vdx - cot\,vdx - vdx - xdv = 0$

$-cotvdx - xdv = 0 \longrightarrow : (-x.cotv)$

$\frac{1}{x}dx + \frac{1}{cotv}dv = 0$

$\int \frac{1}{x}dx + \int \frac{1}{cotv}dv = C1$

$ln\,x\,dx + \int \frac{1}{\cos v/\sin v}dv = C1$

$ln\,x\,dx + \int \frac{\sin v}{\cos v}dv = C1$

$mis: u = cos\,v$

$du = -\sin v\,dv$

$-du = \sin v\,dv$

$ln\,x + \int \frac{\sin v}{\cos v}dv = C1$

$ln\,x + \int \frac{1}{u}.-du = C1$

$ln\,x + \left(-\int \frac{1}{u}du\right) = C1$

$ln\,x + (-\ln u) = C1$

$ln\,x + (-\ln\cos v) = C1; C1 = -\ln C$

$ln\,x - \ln\cos v = -\ln C$

$-ln\cos v = -\ln C - \ln x$

$$-\ln\cos v = -(\ln C + \ln x)$$
$$-\ln\cos v = -\ln C x$$
$$\cos v = Cx;\ v = \frac{y}{x}$$
$$\cos\frac{y}{x} = Cx$$

3. $\frac{dy}{dx} = 1 + \frac{y}{x} - \cos^2\frac{y}{x}$

Penyelesaian :

$\left(1 + \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)\right)dx = dy$.......................................*(1)*

$$\left(1 + \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)\right)dx - dy = 0;$$

Subsitusikan : $v = \frac{y}{x}, y = vx, dy = vdx + xdv$ *ke (1)*

$$(1 + v - \cos^2(v))dx - (vdx + xdv) = 0$$
$$dx + vdx - \cos^2 vdx - vdx - xdv = 0$$
$$(1 - \cos^2 v)dx - x\,dv = 0$$
$$\sin^2 vdx - xdv = 0 \quad \longrightarrow \quad : (x.\sin^2 v)$$
$$\frac{1}{x}dx - \frac{1}{\sin^2 v}dv = 0$$
$$\int\frac{1}{x}dx - \int\frac{1}{\sin^2 v}dv = C$$
$$\ln x - (-\cot v) = C$$
$$\cot v = C - \ln x; \quad subs.\ v = \frac{y}{x}$$
$$\cot\left(\frac{y}{x}\right) = C - \ln x$$

4. $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

Penyelesaian :

$y\,dx = (x + \sqrt{xy})\,dy$...................................*(1)*

$y\,dx - (x + \sqrt{xy})dy = 0 \qquad subs : y = v.x, dy = vdx + xdv$

$$vxdx - \left(x + \sqrt{vx^2}\right)(vdx + xdv) = 0$$
$$vxdx - (x + x\sqrt{v})(vdx + xdv) = 0$$
$$vxdx - vxdx - x^2dv - vx\sqrt{v}dx - x^2\sqrt{v}dv = 0$$
$$-vx\sqrt{v}dx - x^2dv - x^2\sqrt{v}dv = 0$$
$$-vx\sqrt{v}dx - (x^2 + x^2\sqrt{v})dv = 0$$
$$-x\sqrt{v^2}dx.\sqrt{v} - (x^2 + x^2\sqrt{v})dv = 0$$
$$-x\sqrt{v^2.v}\,dx - x^2(1 + \sqrt{v})dv = 0$$

$$-x\sqrt{v^3}dx - x^2\left(1+\sqrt{v}\right)dv = 0$$

$$-x\sqrt{v^3}dx - x^2\left(1+\sqrt{v}\right)dv = 0 \qquad \longrightarrow : x^2\sqrt{v^3}$$

$$\frac{1}{x}dx + \frac{1+\sqrt{v}}{\sqrt{v^3}}dv = 0$$

$$\int \frac{1}{x}dx + \int \frac{1+\sqrt{v}}{\sqrt{v^3}}dv = C$$

$$ln\, x + \int \left(1+v^{\frac{1}{2}}\right)\left(v^{\frac{-3}{2}}\right)dv = C$$

$$ln\, x + \int \left(v^{\frac{-3}{2}}\right)dv + \int v^{-1}dv = C$$

$$ln\, x - 2v^{\frac{-1}{2}} + ln\, v = C$$

$$ln\, x - \frac{2}{\sqrt{v}} + lnv = C\,; \qquad subs.\ v = \frac{y}{x}$$

$ln\, x - \frac{2}{\sqrt{\frac{y}{x}}} + ln\frac{y}{x} = C$,,,,,,,**selesaikan ke eksponen**

5. $(3x^2y + y^3)dx + (x^3 + 3xy^2)dy = 0$

$Penyelesaian:$

$$(3x^2y + y^3)dx + (x^3 + 3xy^2)dy = 0 \qquad :dibagi\ dengan\ x^3$$

$$\left(3\frac{y}{x} + \left(\frac{y}{x}\right)^3\right)dx + \left(1 + 3\left(\frac{y}{x}\right)^2\right)dy = 0;\ \ subs: v = \frac{y}{x}, y = vx, dy = vdx + xdv$$

$$(3v + v^3)dx + (1 + 3v^2)(vdx + xdv) = 0$$

$$3vdx + v^3dx + vdx + xdv + 3v^3dx + 3v^2xdv = 0$$

$$(3v + v^3 + v + 3v^3)dx + (x + 3v^2x)dv = 0$$

$$(4v + 4v^3)dx + (x + 3v^2x)dv = 0$$

$$(4v + 4v^3)dx + x(1 + 3v^2)dv = 0$$

$$4(v + v^3)dx + x(1 + 3v^2)dv = 0 \qquad : dibagi\ dengan\ x(v + v^3)$$

$$\frac{4}{x}dx + \frac{1 + 3v^2}{v + v^3}dv = 0$$

$$4\int \frac{1}{x} + \int \frac{1 + 3v^2}{v + v^3}dv = C1$$

$$mis: u = v + v^3$$

$$du = 1 + 3v^2dv$$

$$\int \frac{1}{u}du = ln\, u = ln(\,v + v^3)$$

$$4\ lnx + ln(v + v^3) = C1; \qquad misalkan\ : C1 = lnC$$

$$4\ lnx + ln(v + v^3) = ln\, C$$

$ln x^4 + ln(v + v^3) = ln\, C$

$ln\, x^4(v + v^3) = ln\, C$

$x^4(v + v^3) = C$

$x^4\left(\frac{y}{x} + \left(\frac{y}{x}\right)^3\right)$ $=C$ $\longrightarrow$ $x^3y + xy^3 = C$

6. $\frac{dy}{dx} = -\frac{2x^2+y^2}{2xy-3y^2}$

Penyelesaian :

$2xy - 3y^2dy = -(2x^2 + y^2)dx$

$2xy - 3y^2dy = -2x^2 - y^2dx$

$2xy - 3y^2dy + (2x^2 + y^2)dx = 0$

$(2x^2 - y^2)dx + (2xy - 3y^2)dy = 0$

$(2x^2 - y^2)dx + (2xy - 3y^2)dy = 0$ $\longrightarrow$ $: x^2$

$\left(2 + \left(\frac{y}{x}\right)^2\right)dx + \left(2\frac{y}{x} + 3\left(\frac{y}{x}\right)^2\right)dy = 0; v = \frac{y}{x}, y = vx, dy = vdx + xdv$

$(2 + v^2)dx + (2v - 3v^2)(vdx + xdv) = 0$

$2dx + v^2dx + (2v^2dx + 2vxdv - 3v^3dx - 3v^2xdv) = 0$

$(2dx + 3v^2dx - 3v^3dx) + (2vx - 3v^2x)dv = 0$

$(2 + 3v^2 - 3v^3)dx + x(2v - 3v^2)dv = 0$

$(2 + 3v^2 - 3v^3)dx + x(2v - 3v^2)dv = 0$ $\longrightarrow$ $: x(2 + 3v^2 - 3v^3)$

$\frac{1}{x}dx + \frac{2v - 3v^2}{2 + 3v^2 - 3v^3}dv = 0$

$\int \frac{1}{x}dx \int \frac{2v - 3v^2}{2 + 3v^2 - 3v^3}dv = C1$

$mis : u = 2 + 3v^2 - 3v^3$

$du = 6v - 9v^2dv$

$du = 3(2v - 3v^2)dv$

$\frac{1}{3}du = 2v - 3v^2$

$\frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}ln\, u + C$

$ln\, x + \frac{1}{3}ln(2 + 3v^2 - 3v^3) = C1$ $\longrightarrow$ *dikali dengan 3*

$3ln\, x + ln(2 + 3v^2 - 3v^3) = 3C1$; $\quad$ *mis :* $3C1 = ln\, C$

$ln\, x^3 + ln(2 + 3v^2 - 3v^3) = lnC$

$ln\, x^3\,(2 + 3v^2 - 3v^3) = lnC;\ v = \frac{y}{x}$

$x^3\left(2 + 3\left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right)^3\right) = C$

$2x^3 + 3xy^2 - 3y^3 = C$

Latihan soal:

Carilah solusi dari persamaan diferensial homogen berikut :

1. $(x+y)dx + xdy = 0$
2. $2xy\,dx + (x^2 - y^2)dy = 0$
3. $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$
4. $\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}$
5. $\frac{dy}{dx} = \frac{xe^{\frac{-y}{x}}+y}{x}$
6. $\frac{dy}{dx} = \frac{2xye^{(\frac{x}{y})^2}}{y^2+y^2e^{\left(\frac{x}{y}\right)^2}+2x^2e^{\left(\frac{x}{y}\right)^2}}$
7. $\frac{dy}{dx} = \frac{x^4+3x^2y^2+y^4}{x^3y}$
8. $(1 + 2e^{\frac{x}{y}})dx + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$
9. $2xyy' - y^2 + x^2 = 0$
10. $y\left(y + xe^{\frac{x}{y}}\right)dx - x^2e^{\frac{x}{y}}dy = 0$

Persamaan Diferensial Dengan Koefisen-koefisien Linier

Bentuk umum:

$$(ax + by + c)dx + (px + qy + r)dy = 0 \;........ \quad (1)$$

1) Bila $c = 0,\ r = 0,$ maka (1) menjadi :

$(ax + by)dx + (px + qy)dy = 0$

adalah P.D. homogen dengan substitusi v = y/x

2) Bila $(px + qy) = k(ax + by)\ atau \frac{a}{p} = \frac{b}{q}$ (k = bil. konstanta)

$(ax + by + c)dx + [k(ax + by) + r]dy = 0$ ......(2)

misalkan $ax + by = z$ atau $a\ dx + b\ dy = dz$

(2) menjadi : $(z + c)dx + (kz + r)\ (\frac{dz - a\,dx}{b}) = 0$

atau $(z + c - \frac{kaz + ra}{b})dx + \frac{kz + r}{b}dz = 0$

(adalah P.D. variabel-variabel terpisah).

3) Bila $\dfrac{a}{p} \neq \dfrac{b}{q}$, $c \neq 0$, $r \neq 0$

I. Dengan mempergunakan variabel baru.

$ax + by + c = u \; dengan \begin{vmatrix} a \, b \end{vmatrix} \neq 0$

$px + qy + r = v \qquad dengan \begin{vmatrix} p \, q \end{vmatrix} \neq 0$

dan didapat :

$a \; dx + b \; dy = du$

$p \; dx + q \; dy = dv$

maka $dx = \dfrac{\begin{vmatrix} du & b \\ dv & q \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}} = \dfrac{q \; du - b \; dv}{aq - bp}$

$dy = \dfrac{\begin{vmatrix} a & du \\ p & dv \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}} = \dfrac{a \; dv - p \; du}{aq - bp}$

masukkan harga-harga dx dan dy ke dalam persamaan (1) didapat persamaan differensial homogen.

$(qu - pv)du + (av - bu)dv = 0$ substitusi $z = \dfrac{u}{v}$

II. Dengan mengambil bentuk-bentuk :

$ax + by + c = 0$ adalah persamaan 2 garis

$px + qy + r = 0$ yang berpotongan

misalkan titik potong kedua garis itu (x_1, y_1)

dengan substitusi :

$X = x - x_1$ atau $x = X + x_1$; $dx = dX$

$Y = y - y_1$ atau $y = Y + y_1$; $dy = dY$

ke dalam persamaan (1) didapat :

$(aX + bY)dX + (pX + qY)dY = 0$

(adalah persamaan differensial homogen,dengan substitusi $v = \dfrac{Y}{X}$).

Contoh soal :

Carilah solusi dari persamaan diferensial berikut :

1. $(7x-3y-7)dx+(3x-7y-3)dy=0$

Penyelesaian :

$(7x-3y-7)dx+(3x-7y-3)dy=0$ ......(1)

mis :

$$\begin{matrix} 7x-3y-7=u \\ 3x-7y-3=v \end{matrix} \quad \text{maka} \quad \begin{matrix} 7\,dx-3\,dy=du \\ 3\,dx-7\,dy=dv \end{matrix}$$

$$dx=\frac{\begin{vmatrix} du & -3 \\ dv & -7 \end{vmatrix}}{\begin{vmatrix} 7 & -3 \\ 3 & -7 \end{vmatrix}}=\frac{-7\,du-3dv}{-40}=\frac{7\,du+3dv}{40}$$

$$dy=\frac{\begin{vmatrix} 7 & du \\ 3 & dv \end{vmatrix}}{\begin{vmatrix} 7 & -3 \\ 3 & -7 \end{vmatrix}}=\frac{7\,dv-3\,du}{-49+9}=\frac{7\,dv-3\,du}{-40}=\frac{3\,du-7\,dv}{40}$$

harga-harga dx dan dy disubstitusi ke persamaan (1) , maka :

$$u\left(\frac{7\,du+3\,dv}{40}\right)+v\left(\frac{-7\,dv+3\,du}{-40}\right)=0 \quad \xrightarrow{\times(-40)}$$

$$u(7\,du+3\,dv)+v(3\,du-7\,dv)=0$$

$$7u\,du+3u\,dv+3v\,du-7v\,dv=0$$

$$(7u+3v)du+(3u-7v)dv=0 \text{ ......(2)}$$

$$substitusi\, u = v.z \rightarrow du = v\,dz + z\,dv$$

$$(7(vx) + 3v)(v\,dz + z\,dv) + (3(vz) - 7v)dv = 0$$

$$7v^2 z\,dz - 7vz^2\,dv - 3v^2\,dz - 3vz\,dv - 3vz\,dv + 7v\,dv = 0$$

$$(7v^2 z + 3v^2)dz + (7vz^2 + 6vz - 7v)dv = 0$$

$$\underline{v^2(7z+3)dz + v(7z^2 + 6z - 7)dv = 0} \xrightarrow{:v^2(7z^2+6z-7)}$$

$$\frac{7z+3}{7z^2+6z-7}dz + \frac{dv}{v} = 0$$

$$\int \frac{7z+3}{7z^2+6z-7}dz + \int \frac{dv}{v} = C_1$$

$$\underline{\frac{1}{2}\ln(7z^2+6z-7) + \ln v = C_1} \xrightarrow{\times(2)}$$

$$\ln(7z^2+6z-7) + \ln v^2 = 2C_1 \qquad ambil\ 2C_1 = \ln C$$

$$\ln(7z^2+6z-7) + \ln v^2 = \ln C$$

$$(7z^2+6z-7)v^2 = C,,,, substitusiskan \rightarrow nilai\ z dan v$$

2. $(x + 2y - 1)dx + (2x - y - 7)dy = 0$

Penyelesaian :

a. Cara I

$(x + 2y - 1)dx + (2x - y - 7)dy = 0$ ......(1)

mis :

$$\begin{matrix} x + 2y - 1 = u \\ 2x - y - 7 = v \end{matrix} \quad \text{maka} \quad \begin{matrix} dx + 2\,dy = du \\ 2\,dx - dy = dv \end{matrix}$$

$$dx = \frac{\begin{vmatrix} du & +2 \\ dv & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{-du - 2dv}{-1-4} = \frac{du + 2dv}{5}$$

$$dy = \frac{\begin{vmatrix} 1 & du \\ 2 & dv \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{dv - 2du}{-5} = \frac{2du - dv}{5}$$

harga-harga dx dan dy dimasukkan dalam persamaan (1) :

$$u\left(\frac{du+2dv}{5}\right)+v\left(\frac{2du-dv}{5}\right)=0 \quad \xrightarrow{\times 5}$$

$u(du+2dv)+v(2du-dv)=0$ (P.D. homogen)

$(u+2v)du+(2u-v)dv=0$ ……(2)

substitusi : $u=v.z$

maka $du=v\,dz+z\,dv$

(2) menjadi :

$(vz+2v)(v\,dz+z\,dv)+(2vz-v)dv=0$

$(v^2z+2v^2)dz+(vz^2+2vz+2vz-v)dv=0$

$$v^2(z+2)dz+v(z^2+4z-1)dv=0 \quad \xrightarrow{:v^2(z^2+4z-1)}$$

$$\frac{z+2}{z^2+4z-1}dz+\frac{dv}{v}=0$$

maka :

$$\int\frac{z+2}{z^2+4z-1}dz+\int\frac{dv}{v}=C_1$$

$$\frac{1}{2}\int\frac{d(z^2+4z)}{z^2+4z-1}+\int\frac{dv}{v}=C_1$$

$$\frac{1}{2}\ln\left(z^2+4z-1)\right)+\ln v=C_1$$

$$\ln(z^2+4z-1)+\ln(v)^2=2C_1$$

$$\ln v^2\left(\frac{u^2}{v^2}+4\frac{u}{v}-1\right)=2C \qquad ambil\, 2C_1=\ln C$$

$u^2+4uv-v^2=C$ ganti harga-harga u dan v

$(x+2y-1)^2+4(x+2y-1)(2x-y-7)-(2x-y-7)^2=C$

$(x^2+4y^2+1+4xy-2x-4y)+(8x^2-4xy-28x+16xy-$

$8y^2-56y-8x+4y+28)+(-4x^2-y^2-49+4xy+28xy-14y)=C_1$

$5x^2+20xy-5y^2-10x-70y-20=C_1$

$x^2+4xy-y^2-2x-14y=C$ $\qquad (C=\frac{1}{5}C_1+4)$

b. Cara II

$(x+2y-1)dx+(2x-y-7)dy=0$ ……(1)

ambil bentuk-bentuk :

$$x + 2y - 1 = 0$$
$$2x - y - 7 = 0$$

maka $x = \dfrac{\begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \dfrac{-1-14}{-1-4} = 3$

$$y = \dfrac{\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \dfrac{7-2}{-5} = -1$$

titik potong kedua garis $(3,-1)$

misal :

$X = x - 3$ maka $x = X + 3 \rightarrow dx = dX$

$Y = y + 1$ $\quad y = Y - 1 \rightarrow dy = dY$

masukkan dalam persamaan (1)

$$(X + 3 + 2Y - 2 - 1)dX + (2X + 6 - Y + 1 - 7)dY = 0$$
$$(X + 2Y)dX + (2X - Y)dY = 0$$

substitusi $Y = vX$

$$(X + 2vX)dX + (2X - vX)(X\,dv + v\,dX) = 0$$
$$(X + 2vX + 2vX - v^2 X)dX + (2X^2 - vX^2)dv = 0$$
$$(X + 4vX - v^2 X)dX + (2X^2 - vX^2)dv = 0$$
$$X(1 + 4v - v^2)dX + X^2(2 - v) = 0 \quad \xrightarrow{:X^2(1+4v-v^2)}$$

$$\frac{dX}{X} + \frac{2 - v}{1 + 4v - v^2}dv = 0$$
$$\int \frac{dX}{X}dX + \int \frac{2 - v}{1 + 4v - v^2}dv = C_1$$
$$\ln X + \frac{1}{2}\ln(1 + 4v - v^2) = C_1 \quad \xrightarrow{\times 2}$$
$$\ln X^2 + \ln(1 + 4v - v^2) = 2C_1 \qquad misalkan 2C_1 = \ln C$$

$\ln X^2(1+4v-v^2)=\ln C$

$X^2(1+4v-v^2)=C$

substitusi $v=\dfrac{Y}{X}$

$X^2(1+4\dfrac{Y}{X}-\dfrac{Y^2}{X^2})=C$

$X^2+4XY-Y^2=C$

substitusi $X=x-3$ *dan* $Y=y+1$

maka menghasilkan : $x^2-y^2+4xy-2x-14y-4=C$

3. $(x-2y+9)dx-(3x-6y+19)dy=0$

Penyelesaian :

$(x-2y+9)dx-(3x-6y+19)dy=0$ ......(1)

karena $3x-6y=3(x-2y)$ dalam hal ini $k=3$

(1) boleh ditulis :

$(x-2y+9)dx-[3(x-2y)+19]dy=0$ ......(2)

misalkan :

$x-2y=z$

$dx-2dy=dz$

$-dy=\dfrac{dz-dx}{2}$

$dy=\dfrac{dx-dz}{2}$

persamaan (2) menjadi :

$(z+9)dx-(3z+19)(\dfrac{dx-dz}{2})=0$

$(z+9-\dfrac{3}{2}z-\dfrac{19}{2})dx-(-\dfrac{3}{2}z-\dfrac{19}{2})dz=0$

$(-\dfrac{1}{2}z-\dfrac{1}{2})dx-(-\dfrac{3}{2}z-\dfrac{19}{2})dz=0$

$-\dfrac{1}{2}(z+1)dx+\dfrac{1}{2}(3z+19)dz=0$ $\longrightarrow :(-\frac{1}{2})(z+1)$

$$dx - \frac{(3z+19)}{z+1}dz = 0$$

$$\int dx - \int (3 + \frac{16}{z+1})dz = C$$

$$x - 3z - 16\ln z + 1 = C$$

$$x = 3z + 16\ln z + 1 + C$$

4. $$\frac{dy}{dx} = \frac{x+y+1}{4x+4y+1}$$

Penyelesaian :

$$\frac{dy}{dx} = \frac{x+y+1}{4x+4y+1} \quad \text{...(1)}$$

$$(x+y+1)dx = (4x+4y+1)dy$$

$$(x+y+1)dx - (4x+4y+1)dy = 0 \text{ ......(2)}$$

$$4x+4y = 4(x+y) \qquad dalamhal ini\, k = 4$$

$$(x+y+1)dx - [4(x+y)+1]dy = 0$$

misalkan :

$$x + y = z$$

$$dx + dy = dz$$

$$dy = dz - dx$$

$$(z+1)dx - (4z+1)(dz - dx) = 0$$

$$(z+1-4z-1)dx - (4z+1)dz = 0$$

$$(z+1-4z-1)dx - (4z+1)dz = 0$$

$$\underline{-3z\,dx + (4z+1)dz = 0} \longrightarrow :z$$

$$-3\,dx + \frac{4z+1}{z}dz = 0$$

$$-3\int dx + \int (\frac{4z+1}{z})dz = C$$

$$-3x + 4z + \ln z = C$$

$$substitusi \;\; z = x + y$$

$$-3x + 4x + 4y + \ln(x+y) = C$$

$$x + 4y + \ln(x+y) = C$$

$$(x+y+1)dx + (2x+2y+1)dy = 0$$

5. $(x+y+1)dx+(2x+2y+1)dy=0$

Penyelesaian :

$(x+y+1)dx+(2x+2y+1)dy=0$ ....(1)

ternyata : $2x+2y=2(x+y)$; (dalam hal ini k = 2)

(1) boleh ditulis :

$(x+y+1)dx+[2(x+y)+1]dy=0$ ....(2)

persamaan (2) menjadi :

$(z+1)dx+(2z+1)(dz-dx)=0$

$(z+1-2z-1)dx+(2z+1)dz=0$

$-z\,dx+(2z+1)dz=0 \quad :z$

$-dx+\frac{2z+1}{z}dz=0$

$-dx+(2+\frac{1}{z})dz=0$

maka :

$-\int dx+\int(2+\frac{1}{z})dz=C$

$-x+2z+\ln z=C$ ganti $z=x+y$

$-x+2x+2y+\ln(x+y)=C$

$x+2y+\ln(x+y)=C$

Latihan soal

Selesaikan persamaan diferensial koefisen linear berikut:

1. $\frac{dy}{dx}=\frac{6x-2y-7}{2x+3y-6}$

2. $(3x-2y+1)dx-(6x-4y+1)dy=0$

3. $(7x-3y-7)dx+(3x-7y-3)dy=0$

4. $\frac{dy}{dx}=\frac{x-y-4}{x+y-2}$

5. $(2x-y+3)dx+\ (4x-2y+7)dy=0$

6. *(10x-15y+5)dx+(2x-3y+1)dy=0*

7. $\frac{dy}{dx}=\frac{1-2y+4x}{1+y+2x}$

8. $y'=\frac{y-x+1}{y-x+5}$

9. $y'=\frac{2x+y+3}{x+2y+9}$

10. $(3y - x)\frac{dy}{dx} = (3x - y + 4)$

Persamaan Diferensial Exact

Bentuk umum :

$$P(x,y)dx + Q(x,y)dy = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.(1)$$

Persamaan (1) disebut persamaan differensial exact jika ruas kiri adalah differensial dari $\mathbf{f(x,y) = 0}$

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.(2)$$

Dengan membandingkan persamaan (1) dan (2), terlihat bahwa persamaan (1) bersifat pasti (exact) jika ada suatu fungsi f(x,y) yang bersifat :

$$\left.\begin{aligned} P &= \left(\frac{\partial f}{\partial x}\right) \\ Q &= \left(\frac{\partial f}{\partial y}\right) \end{aligned}\right\} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.(3)$$

Jika fungsi-fungsi P dan Q terdefinisikan dan terdiferensial di semua titik pada bidang xy dalam kurva tertutup dan tidak memotong kurva fungsi itu sendiri, maka dari persamaan (3) diperoleh:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} \quad \text{dan} \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} \qquad .\;.\;.\;.\;(4)$$

Terlihat bahwa dua turunan kedua di atas adalah sama, sehingga $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ adalah syarat perlu dan syarat cukup agar $P\,dx + Q\,dy = 0$ merupakan persamaan differensial exact.

- **Menentukan Penyelesaian Persamaan Differensial Exact**

Fungsi f(x, y) sebagai fungsi penyelesaian persamaan differensial exact diperoleh melalui operasi pengintegralan sebagai berikut:

a. Integralkan terhadap variabel *x*, sehingga diperoleh:

$$f(x,y) = \int P\,dx + C(y)$$

$C(y)$: fungsi pengintegralan dan nilainya dapat ditentukan dengan $\frac{\partial f}{\partial y} = Q$

b. Integralkan terhadap variabel *y*, sehingga diperoleh:

$$f(x,y) = \int Q\,dy + C(x)$$

$C(x)$: fungsi pengintegralan dan nilainya dapat ditentukan dengan $\frac{\partial f}{\partial x} = P$

Contoh Soal

Perlihatkan bahwa persamaan differensial berikut adalah exact dan tentukan penyelesaiannya!

1. $(x + y\cos x)dx + (\sin x)\, dy = 0$

Penyelesaian:

$(x + y\cos x)dx + (\sin x)\, dy = 0\ \ldots . (1)$

$P = x + y\cos x \qquad Q = \sin x$

$\frac{\partial P}{\partial y} = \cos x \;;\quad \frac{\partial Q}{\partial x} = \cos x$ ternyata $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \cos x$

Maka persamaan (1) adalah Persamaan Differensial Exact

$f(x, y) = \int P\, dx + C(y)$

$f(x, y) = \int (x + y\cos x)dx + C(y)$

$f(x, y) = \frac{1}{2}x^2 + y\sin x + C(y)\ \ldots\ldots (2)$

$\frac{\partial f}{\partial y} = \sin x + C'(y)$

Berdasarkan bentuk umum, maka nilai C(y) adalah

$Q = \frac{\partial f}{\partial y}$

$\sin x = \sin x + C'(y)$

$C'(y) = 0 \qquad \int C'(y) = \int 0\, dy$

$C(y) = C_1$

Kita substitusikan nilai C(y) pada persamaan 2

$f(x, y) = \frac{1}{2}x^2 + y\sin x + C_1 = 0$

$f(x, y) = \frac{1}{2}x^2 + y\sin x = -C_1$ misalkan $-C_1 = C$

Jadi: $f(x, y) = \frac{1}{2}x^2 + y\sin x = C$

2. $(2x + 3y)dx + (3x + 4y)dy = 0$

Penyelesaian :

$(2x + 3y)dx + (3x + 4y)dy = 0\ \ldots\ldots (1)$

$P = 2x + 3y \qquad Q = 3x + 4y$

$\frac{\partial P}{\partial y} = 3 \;;\; \frac{\partial Q}{\partial x} = 3$ ternyata $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 3$

Maka persamaan (1) adalah Persamaan Diferensial Exact

$f(x, y) = \int P\, dx + C(y)$

$f(x, y) = \int (2x + 3y)dx + C(y)$

$f(x, y) = x^2 + 3xy + C(y)\ \ldots\ldots\ldots (2)$

$\frac{\partial f}{\partial y} = 3x + C'(y)$

Berdasarkan bentuk umum, maka nilai C(y) adalah

$Q = \frac{\partial f}{\partial y}$

$3x + 4y = 3x + C'(y)$

$C'(y) = 4y$

$\int C \quad (y) = \int 4y\, dy$

$C(y) = 2y^2 + C_1$

Kita substitusikan nilai C(y) pada persamaan (2)

$f(x, y) = x^2 + 3xy + 2y^2 + C_1 = 0$

$f(x, y) = x^2 + 3xy + 2y^2 = -C_1$ misalnya: - C_1 = C

Jadi: $f(x, y) = x^2 + 3xy + 2y^2 = C$

Penyelesaian cara 2:

$f(x, y) = \int Q\, dy + C(x)$

$f(x, y) = \int (3x + 4y)dy + C(x)$

$f(x, y) = 3xy + 2y^2 + C(x)$................(3)

$\frac{\partial f}{\partial x} = 3y + C'(x)$

Berdasarkan bentuk umum, maka nilai C(y) adalah

$P = \frac{\partial f}{\partial x}$

$2x + 3y = 3y + C'(x)$

$C'(x) = 2x$

$\int C'(x) = \int 2x\ dx$

$C(x) = x^2 + C_1$

Kita substitusikan nilai C(x) pada persamaan (3)

$f(x, y) = 3xy + 2y^2 + x^2 + C_1 = 0$

$3xy + 2y^2 + x^2 = -C_1$ misalkan: - C_1 = C

Jadi: $3xy + 2y^2 + x^2 = C$

3. $(15x^2y^2 - y^4)dx + (10x^3y - 4xy^3 + 5y^4)dy = 0$

Penyelesaian:

$(15x^2y^2 - y^4)dx + (10x^3y - 4xy^3 + 5y^4)dy = 0 \ldots\ldots (1)$

$P = 15x^2y^2 - y^4$ $Q = 10x^3y - 4xy^3 + 5y^4$

$\frac{\partial P}{\partial y} = 30x^2y - 4y^3$; $\frac{\partial Q}{\partial x} = 30x^2y - 4y^3$

ternyata $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 30x^2y - 4y^3$

Maka persamaan (1) adalah Persamaan Differensial Exact

$f(x, y) = \int P\, dx + C(y)$

$f(x, y) = \int 15x^2y^2 - y^4 dx + C(y)$

$f(x, y) = 5x^3y^2 - xy^4 + C(y)\ \ldots\ldots (2)$

$$\frac{\partial f}{\partial y} = 10x^3y - 4xy^3 + C'(y)$$

Berdasarkan bentuk umum, maka nilai C(y) adalah

$$Q = \frac{\partial f}{\partial y}$$

$$10x^3y - 4xy^3 + 5y^4 = 10x^3y - 4xy^3 + C'(y)$$

$$C'(y) = 5y^4$$

$$\int C'(y) = \int 5y^4\, dy$$

$$C(y) = y^5 + C_1$$

Kita substitusikan nilai C(y) pada persamaan 2

$$f(x,y) = 5x^3y^2 - xy^4 + y^5 + C_1 = 0$$

$5x^3y^2 - xy^4 + y^5 = -C_1$ misalnya: $-C_1=C$

Jadi: $5x^3y^2 - xy^4 + y^5 = C$

4. $e^{x^2y}(1 + 2x^2y)dx + x^3e^{x^2y}\, dy = 0$

Penyelesaian :

$P = e^{x^2y}(1 + 2x^2y)$ $\qquad$ $Q = x^3e^{x^2y}$

syarat :

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

$$\frac{\partial\left(e^{x^2y}(1 + 2x^2y)\right)}{\partial y} = \frac{\partial\left(x^3e^{x^2y}\right)}{\partial x}$$

$$\frac{\partial\left(e^{x^2y}(1 + 2x^2y)\right)}{\partial y} = x^2e^{x^2y}(1 + 2x^2y) + e^{x^2y}2x^2$$

$$= x^2e^{x^2y} + 2x^4ye^{x^2y} + 2x^2e^{x^2y}$$

$$= 3x^2e^{x^2y} + 2x^4ye^{x^2y}$$

$Q = x^3e^{x^2y}$

$$\frac{\partial Q(x,y)}{\partial x} = 3x^2e^{x^2y} + x^3 2xye^{x^2y}$$

$= 3x^2e^{x^2y} + 2x^4ye^{x^2y}$ **merupakan PD. Exact**

$P = \frac{\partial f}{\partial x} = e^{x^2y}(1 + 2x^2y)$ $\qquad$ $Q = \frac{\partial f}{\partial y} = x^3e^{x^2y}$

❖ $f(x,y) = \int x^3e^{x^2y}dy + C(x)$

$$= x^3\frac{1}{x^2}e^{x^2y} + C(x)$$

$$= xe^{x^2y} + C(x)$$

$$\frac{\partial f(x,y)}{\partial x} = e^{x^2y} + 2xye^{x^2y}.x + C'(x)$$

$$= e^{x^2y} + 2x^2ye^{x^2y} + C'(x)$$

$$\frac{\partial f(x\,,y)}{\partial x} = Q$$
$$e^{x^2y} + 2x^2ye^{x^2y} + C'(x) = e^{x^2y}\,(1 + 2x^2y)$$
$$e^{x^2y}\,(1 + 2x^2y) + C'(x) = e^{x^2y}\,(1 + 2x^2y)$$
$$C'(x) = 0$$
$$C(x) = \mathrm{C}_1$$

$$Jadi, f(x\,,y) = xe^{x^2y} + \mathrm{C}_1 = 0$$
$$xe^{x^2y} = -\mathrm{C}_1 \quad (\text{mis:} -\mathrm{C}_1 = \mathrm{C})$$
$$xe^{x^2y} = \mathrm{C}$$

❖ $f(x,y) = \int e^{x^2y}(1 + 2x^2y)dx$

$misalkan : U = e^{x^2y}$ $\qquad dv = dx$

$du = 2xye^{x^2y}dx$ $\qquad V = x$

$$\int Udv = e^{x^2y}.x - \int x.2xye^{x^2y}\,dx$$
$$= xe^{x^2y} - 2x^2ye^{x^2y}dx + \int 2x^2\,ye^{x^2y}dx$$
$$= xe^{x^2y} + C(y)$$
$$\frac{\partial f}{\partial y} = x\,xe^{x^2y} + C'(y)$$

$$\frac{\partial f}{\partial y} = Q$$
$$x\,xe^{x^2y} + C'(y) = x^3e^{x^2y}$$
$$C'(y) = 0$$
$$C(y) = C_1$$

$$Jadi, f(x\,,y) = xe^{x^2y} + \mathrm{C}_1 = 0$$
$$xe^{x^2y} = -\mathrm{C}_1 \quad (\text{mis:} -\mathrm{C}_1 = \mathrm{C})$$
$$xe^{x^2y} = \mathrm{C}$$

5. $(3y^2 + y\sin 2xy)dx + (6xy + x\sin 2xy)dy = 0$

Penyelesaian :

$P = 3y^2 + y\sin 2xy$ $\qquad Q = 6xy + x\sin 2xy$

syarat :

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$
$$\frac{\partial P(x,y)}{\partial y} = 6y + \sin 2xy + 2xy\cos 2xy$$
$$\frac{\partial Q(x,y)}{\partial x} = 6y + \sin 2xy + 2xy\cos 2xy$$

merupakan PD. Exact

$P = \frac{\partial f}{\partial x} = 3y^2 + y \sin 2xy \qquad Q = \frac{\partial f}{\partial y} = 6xy + x \sin 2xy$

❖ $f(x,y) = \int (3y^2 + y \sin 2xy)\, dx + C(y)$

$= 3xy^2 - \frac{y}{2y} \cos 2xy + C(y)$

$= 3xy^2 - \frac{1}{2} \cos 2xy + C(y)$

$$\frac{\partial f(x,y)}{\partial y} = 6xy - \left(-\frac{2x}{2} \sin 2xy\right) + C'(y)$$

$= 6xy + x \sin 2xy + C'(y)$

$\frac{\partial f(x,y)}{\partial y} = Q$

$6xy + x\sin 2xy + C'(y) = 6xy + x\sin 2xy$

$C'(y = 0$

$C(y) = C_1$

$$Jadi, f(x\,,y) = 3xy^2 - \frac{1}{2} \cos 2xy + C_1 = 0$$

$3xy^2 - \frac{1}{2} \cos 2xy = -C_1$

$6xy^2 - \cos 2xy = -2C_1 \quad (mis: -2C_1 = C\,)$

$6xy^2 - \cos 2xy = C$

❖ $f(x,y) = \int (6xy + x\sin 2xy\,)\ dy + C(x)$

$$= 3xy^2 - \frac{x}{2x} \cos 2xy + C(x)$$

$$= 3xy^2 - \frac{1}{2} \cos 2xy + C(x)$$

$$\frac{\partial f(x,y)}{\partial y} = 3y^2 — \left(-\frac{2y}{2} \sin 2xy\right) + C'(x)$$

$= 3y^2 + y \sin 2xy + C'(x)$

$$\frac{\partial f(x,y)}{\partial y} = P$$

$3y^2 + y \sin 2xy + C'(x) = 3y^2 + y \sin 2xy$

$C'(x) = 0$

$C(x) = C_1$

$$\text{Jadi}\,, f(x,y) = 3xy^2 - \frac{1}{2} \cos 2xy + C_1 = 0$$

$$3xy^2 - \frac{1}{2} \cos 2xy = -C_1$$

$6xy^2 - \cos 2xy = -2C_1 \quad (mis: -2C_1 = C\,)$

$6xy^2 - \cos 2xy = C$

Latihan Soal:

Perlihatkan bahwa persamaan differensial berikut adalah exact dan tentukan penyelesaiannya!

1. $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$
2. $(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0)$
3. $3xy\sqrt{(1 + x^2)}\,dx + \left[\sqrt{(1 + x^2)}^3 + \sin y\right] dy = 0$
4. $\frac{x\,dx}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y\,dy}{(x^2+y^2)^{\frac{3}{2}}} = 0$
5. $(\frac{2x}{y} + 5y^2 - 4x)\,dx + (3y^2 - \frac{x^2}{y^2} + 10xy)\,dx = 0$
6. $(2xy^2 + 2xye^{2x} + e^{2x}y)dx + (2x^2y + xe^{2x})dy = 0$
7. $(3x^2 + 3xy^2)\,dx + (3x^2y - 3y^2 + 2y)dy = 0$
8. $(2x\cos y - e^x)dx - x^2 \sin y\,dy = 0$
9. $(y^2 + 6x^2\,y)\,dx + (\,2xy + 2x^3\,)\,dy = 0$
10. $(\,y + 3x\,)\,dx + x\,dy = 0$

Faktor Integrasi

Bentuk Umum:

Jika $P(x,y)dx + Q(x,y)dy = 0$ ……………….(1)

tidak exact, maka kita selalu bisa menjadikannya persamaan differensial exact, dengan memperbanyak pers. (1) dengan suatu fungsi u(x,y), sehingga:

$u\,P\,dx + u\,Q\,dy = 0$ …………………………..(2)

adalah persamaan differensial exact maka berlakulah:

$$\frac{\partial(u\,P)}{\partial y} = \frac{\partial\,(u\,Q)}{\partial x}$$

atau $\frac{u\,\partial P}{\partial y} + \frac{P\,\partial u}{\partial x} = \frac{u\,\partial Q}{\partial x} + \frac{Q\,\partial u}{\partial y}$

$$u\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial u}{\partial x} - P\frac{\partial u}{\partial y}$$

Dari persamaan ini harga u dapat dicari, dan setelah itu harga u dimasukkan dalam persamaan (2) terjadilah P.D. exact

Pada penyelesaian soal-soal mengenai faktor integrasi, ada 2 hal yang harus diperhatikan :

1. Pada soal-soal mengenai faktor integrasi, telah dicantumkan/ditentukan jenis dari faktor-faktor integrasinya.
   misalnya : mempunyai faktor integrasi hanya fungsi dari x,
   mempunyai faktor integrasi tergantung dari y
   atau u(y) , u(x,y), $u(x^2 + y^2)$, dan lain-lain

2. Pada soal-soal mengenai faktor integrasi, tidak dicantumkan/ditentukan jenis dari faktor-faktor integrasinya, hingga kita harus mencari sendiri, jenis apa faktor integrasi yang sesuai dengan soal-soal.

Dalam hal A, kita boleh langsung masukkan jenis dari faktor integrasinya kedalam persamaan.

$$u\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial u}{\partial x}-P\frac{\partial u}{\partial y}$$

Sehingga harga dari faktor integrasi u dapat diperoleh.

Dalam hal B, kita belum mempunyai rumus yang tepat untuk langsung memperoleh *jenis* dan *harga* dari faktor integrasi yang sesuai. Walaupun demikian kita masih bias mencari *jenis* dan *harga* dari faktor integrasi dengan memperhatikan ketentuan-ketentuan yang tercantum dibawah ini.

- Bila faktor integrasinya hanya tergantung dari x maka :

  u = u(x), $\frac{\partial u}{\partial x}=\frac{du}{dx}\ dan\ \frac{\partial u}{\partial y}=0$

  maka rumus faktor integrasi menjadi :

  $u(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})=Q\frac{du}{dx}-0$

  atau $\frac{du}{u}=\frac{(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})}{Q}dx$

  $$\int\frac{du}{u}=\int\frac{(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})}{Q}dx$$

  $$\ln u=\int\frac{(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})}{Q}dx$$

  $$u=e^{\int\frac{(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})}{Q}dx}$$

- Bila faktor integrasinya hanya tergantung dari y.

  maka u = u(y), $\frac{\partial u}{\partial x}$ = 0 dan$\frac{\partial u}{\partial y}=\frac{du}{dy}$

  maka rumus faktor integrasi menjadi

  $u(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})=0-P\frac{du}{dy}$

  atau $\frac{du}{u}=\frac{(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})}{-P}dy$, **diselesaikan sama seperti apabila tergantung dari x**

- Bila faktor integrasinya hanya tergantung dari (x ± y).

  maka u = u(z)

  =(x ± y)

  $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}=u'(z).\frac{dz}{dx}=u'(z)$

  $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}.\frac{\partial z}{\partial y}=u'(z).\frac{dz}{dy}=\pm u'(z)$$

  maka rumus factor integrasi menjadi

$u(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) =$ Q $.u'(z)$ – P $.u'(z)$

= (Q ∓ P) $.u'(z)$

atau $\frac{u'(z)}{u} = \frac{(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})}{Q \mp P}$ (jadikan fungsi z)

atau $\frac{du}{u} = \frac{(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})}{Q \mp P}$dz ,**diselesaikan sama seperti apabila tergantung dari x**

- Bila faktor integrasi hanya tergantung dari (x y) atau u = u(z) = u(xy)

  maka $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} . \frac{\partial z}{\partial x} = u'(z). \frac{dz}{dx} = u'(z)$. y

  $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} . \frac{\partial z}{\partial y} = u'(z). \frac{dz}{dy} = u'(z)$. x

  maka rumus faktor integrasi menjadi

  $u(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = Q\ y\ u'(z) -$ P . x $u'(z)$

  = (Q y – P x) $.u'(z)$

  maka $\frac{u'(z)}{u} = \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{Q\ y - P\ x}$ (jadikan fungsi dari z)

  atau $\frac{du}{u} = \frac{(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})}{Q\ y - P\ x} dz$, **diselesaikan sama seperti apabila tergantung dari x**

- Bila faktor integrasi hanya tergantung dari ($x^2 + y^2$ atau u = u(z) = u($x^2 + y^2$).

  maka $\frac{\partial u}{\partial x} = = \frac{\partial u}{\partial z} . \frac{\partial z}{\partial x} = u'(z). \frac{dz}{dx} = u'(z)$. 2x

  $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} . \frac{\partial z}{\partial y} = u'(z). \frac{dz}{dy} u'(z)$. 2y

  maka rumus faktor integrasi menjadi :

  $u(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) =$ Q $.u'(z)$. 2x – P $.u'(z)$. 2y

  = (2x. Q – 2y .P) $.u'(z)$

  atau $\frac{u'(z)}{u} = \frac{(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})}{2x.Q - 2y.P}$ (jadikan fungsi dari z)

  $\frac{du}{u} = \frac{(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})}{2x.Q - 2y.P}$ dz ,**diselesaikan sama seperti apabila tergantung dari x**

Contoh Soal:

1. Tentukan lah persamaan differensial dari (x + y)dx + dy = 0 yang mempunyai faktor integrasi hanya fungsi dari x.

Penyelesaian :

(x + y)dx + dy = 0 . . . . (1)

Faktor integrasi u = u(x) ;

Maka $\frac{\partial u}{\partial x} = \frac{du}{dx}, \frac{\partial u}{\partial y} = 0$

$u = e^{\int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} dx}$

$u = e^{\int \frac{1-0}{0} dx}$

$u = e^{\int dx}$

$u = e^{x}$

Persamaan (1) dikalikan dengan e^{x}

e^{x} (x + y)dx + e^{x} dy = 0 . . . . (2)

P′ = $e^{x}(x + y)$ $\frac{\partial P\prime}{\partial y} = e^{x}$

Q′ = e^{x} $\frac{\partial Q\prime}{\partial x} = e^{x}$

$\frac{\partial P\prime}{\partial y} = \frac{\partial Q\prime}{\partial x}$ maka PD diatas sudah exact

Tentu f(x,y) = $\int e^{x}\, dy + C(x)$

$= ye^{x} + C(x)$

Sehingga di dapatlah P′ = $\frac{\partial f}{\partial x} = ye^{x}$ + C′(x) = $xe^{x} + ye^{x}$

Maka C′(x) = xe^{x}

C(x) = $\int x\, e^{x}$ dx = $\int x\, de^{x}$

= $xe^{x} - e^{x}$ + C_1

Maka solusi umum dari PD f(x,y) = 0

f(x, y) = $e^{x}y + xe^{x} - e^{x}$ + C_1 = 0 , mis: - C_1= C

$e^{x}y + xe^{x} - e^{x}$ = C

2. Tentukanlah persamaan differensial dari (12x^2y + 3xy^2 + 2y)dy + (6x^3 + 3x^2y + 2x) dx = 0 yang mempunyai faktor integrasi dari (x + y)

Penyelesaian :

$\frac{\partial P}{\partial y}$ = 12x^2 + 6xy + 2 $\frac{\partial Q}{\partial x}$ = 18x^2 + 6xy + 2

Misalkan : z = x y

U = u(z) = u (x y)

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}$ = u′(z) · y

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y}$ = u′(z) · x

$u = e^{\int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{2xQ - 2yQ}}$

$\mathrm{u} = \mathrm{e}^{\int \frac{[(12x^2 + 6xy + 2) - (18x^2 + 6xy + 2)]}{(6x^3y + 3x^{2y^2} + 2xy) - (12x^3y + 3x^2y^2 + 2xy)} dz}$

$u = e^{\int \frac{-6x^2}{-6x^3y} dz}$

$u = e^{\int \frac{1}{xy} dz}$, mis xy = z

$u = e^{\int \frac{1}{z} dz}$

$u = e^{\ln z}$, jadi $u = z = xy$

u = xy substitusi u ke persamaan umum

$\mathrm{xy}\ (12x^2\mathrm{y} + 3xy^2 + 2\mathrm{y})\mathrm{dy} + \mathrm{xy}(6x^3 + 3x^2y + 2\mathrm{x})\mathrm{dx} = 0$

$(12x^3y^2 + 3x^2y^3 + 2\mathrm{xy}^2)\mathrm{dy} + (6x^4y + 3x^3y^2 + 2x^2\mathrm{y})\mathrm{dx} = 0$

$\frac{\partial P'}{\partial y} = 24x^3y + 9x^2y^2 + 4\mathrm{xy}$

$\frac{\partial Q'}{\partial x} = 24x^3y + 9x^2y^2 + 4\mathrm{xy}$

$\frac{\partial P'}{\partial y} = \frac{\partial Q'}{\partial x}$ → Terbukti Exact

Selesaikan dengan persamaan exact

f (x,y) = $\int P'$dx + C (x)

dx + C (x)

dx + C (y)

= $\int (12x^3y^2 + 3x^2y^3 + 2\mathrm{xy}^2) + \mathrm{C(y)}$

= $3x^4y^2 + x^3y^3 + x^2y^2 + \mathrm{C(y)}$

$\frac{\partial f}{\partial y} = 6x^4y + 3x^3y^2 + 2x^2y + C(y)$

$\frac{\partial f}{\partial y} = Q$

$6x^4y + 3x^3y^2 + 2x^2\mathrm{y} + \mathrm{C'(y)} = 6x^4y + 3x^3y^2 + 2x^2\mathrm{y}$

$\mathrm{C'(y)} = 0$

$\mathrm{C(y)} = C_1$

f (x,y) = $3x^4y^2 + x^3y^3 + x^2y^2 = \mathrm{C}$

$x^2y^2(3x^2 + \mathrm{xy} + 1) = \mathrm{C}$

Maka solusi umum dari PD nya :

x^2y^2 $(3x^2 + \mathrm{xy} + 1 = \mathrm{C}$

3. Tentukan solusi umum PD berikut, dengan terlebih dahulu menentukan faktor integrasinya $(x^2 + y^2 + x)dx + xydy = 0$

Penyelesaian :

$(x^2 + y^2 + x)dx + xydy = 0$ .... (1)

$P = x^2 + y^2 + x$ $\qquad \frac{\partial P}{\partial y} = 2y$

Q = xy $\frac{\partial Q}{\partial x} = $ y

$$\frac{\partial P}{\partial y} \neq \frac{\partial\, Q}{\partial x}$$

$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{2y - y}{xy} = \frac{y(2-1)}{y(x)} = \frac{1}{x}$ → Fungsi (x) saja

Tentukan harga **u** :

$u = e^{\int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} dx}$

$u = e^{\int \frac{1}{x} dx}$

$u = e^{\ln x}$

u = x

Substitusi u ke persamaan (1) dalam bentuk u P (x) + u Q (y) =0

x (x^2 + y^2 +x)dx + x (xy)dy = 0

(x^3 + xy^2 +x^2)dx + x^2ydy =0

$\frac{\partial P\prime}{\partial y}$ = 2xy $\frac{\partial Q\prime}{\partial X}$ = 2xy → Terbukti Exact

Selesaikan dengan cara PD Exact

f(x,y) = $\int Q' dy + C(x)$

$= \int x^2 y\, dy + C(x)$

$= \frac{1}{2}x^2y^2 + C(x)$

$\frac{\partial f}{\partial x} = P'$

$xy^2 + C'(x) = x^3 + xy^2 + x^2$

$\int C(x) = x^3 + x^2$

$C(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 = C$

Maka solusi umum dari PD F(x,y) = C

$\frac{1}{2}x^2y^2 + \frac{1}{4}x^4 + \frac{1}{3}x^3 = C$

4. Tentukan solusi umum PD berikut, dengan terlebih dahulu menentukan faktor integrasinya (xy^2 + y) dx – x dy = 0

Penyelesian :

(xy^2 + y) dx – x dy = 0 . . . . (1)

P = xy^2 + y $\frac{\partial P}{\partial y} = 2xy + 1$

$Q = -x$ $\frac{\partial Q}{\partial x} = -1$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ = (2xy + 1) – (-1) = 2xy + 1

$$\frac{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}{-P} = \frac{(2xy+1)}{-(xy^2-y)} = \frac{2(xy+1)}{-y(xy+1)} = -\frac{2}{y} \quad \rightarrow \text{Fungsi y saja}$$

Tentukan harga u

$$u = e^{\int \frac{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}{-P} dy}$$

$$u = e^{\int -\frac{2}{y} dy}$$

$$u = e^{-2\ln y}$$

$$u = lny^{-2}$$

$$u = \frac{1}{y^2}$$

substitusi $\frac{1}{y^2}$ ke persamaan 1

$$\frac{1}{y^2}(xy^2 + y)dx - \frac{1}{y^2}(x)\,dy = 0$$

$$\frac{xy^2}{y^2} + \frac{y}{y^2}dx - \frac{x}{y^2}dy = 0$$

$$(x+\frac{1}{y})dx - \frac{x}{y^2}dy \rightarrow \quad P' = x + \frac{1}{y} \quad , \quad \frac{\partial P'}{\partial y} = -y^{-2}$$

$$\text{dan } Q' = -\frac{x}{y^2} \quad , \quad \frac{\partial Q'}{\partial x} = -\ y^{-2}$$

$$f(x,y) = \int Q'\ dy + C(x)$$

$$= \int -\frac{x}{y^2}\, dy + C(x)$$

$$= -\int \frac{x}{y^2}\, dy + C(x)$$

$$= xy^{-1} + C(x)$$

$$\frac{\partial f}{\partial x} = P'$$

$$y^{-1} + C'(x) = x + \frac{1}{y}$$

$$\int C'(x) = x$$

$$C(x) = \frac{1}{2}x^2$$

Maka solusi PD nya ialah $f(x,y) = 0 \rightarrow xy^{-1} + \frac{1}{2}x^2 = C$

5. Tentukan solusi umum PD berikut, dengan terlebih dahulu menentukan faktor integrasinya $(4x^2 + 2xy + 2y)dx + (2x^2 + x + 3y)dy = 0$

Penyelesaian :

$(4x^2 + 2xy + 2y)dx + (2x^2 + x + 3y)dy = 0$ ....(1)

$P = 4x^2 + 2xy + 2y$ $\quad \frac{\partial P}{\partial y} = 2x + 2$

$Q = 2x^2 + x + 3y$ $\quad \frac{\partial Q}{\partial x} = 4x + 1$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ = (2x +2) – (4x + 1) = –2x + 1

Selanjutnya kita pilih pembaginya, yaitu Q – P sehingga di peroleh

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q-P} = \frac{-2x+1}{(2x^2+x+3y)-(4x^2+2xy+2y)}$$

$$= \frac{-2x+1}{-2x^2+x+y-2xy}$$

$$= \frac{-2x+1}{(-2x^2+x)+(-2xy+y)}$$

$$= \frac{-2x+1}{x(-2x+1)+y(-2x+1)}$$

$$= \frac{-2x+1}{(-2x+1)(x+y)}$$

$= \frac{1}{x+y}$ → Fungsi dari (x + y) saja

Misalkan z = x+ y

$$\frac{\partial u}{u} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q-P} \partial z$$

$\frac{\partial u}{u} = \frac{1}{x+y} \partial z$ → $\frac{1}{z} \partial z$

$$\frac{\partial u}{u} = \frac{\partial z}{z}$$

$$\int \frac{\partial u}{u} = \int \frac{\partial z}{z}$$

ln u = ln z

u = z

u = x + y

Substitusi u kepersamaan 1 dalam bentuk u P (x,y)dx + u Q (x,y)dy = 0

(x + y) $(4x^2 + 2xy + 2y)$dx + (x+y) $(2x^2 + x + 3y)$dy = 0

$(4x^3 + 2x^2y + 2xy + 4x^2y + 2xy^2 + 2y^2)$dx + $(2x^3 + x^2 + 3xy + 2x^2y + xy + 3y^2)$dy = 0

$(4x^3 + 6x^2y + 2xy + 2xy^2 + 2y^2)$dx + $(2x^3 + x^2 + 4xy + 2x^2y + 3y^2)$ = 0

Maka dari persamaan diatas kita misalkan lah

$P' = 4x^3 + 6x^2y + 2xy + 2xy^2 + 2y^2$ $\frac{\partial P'}{\partial y} = 6x^2 + 2x + 4xy + 4y$

$Q' = 2x^3 + x^2 + 4xy + 2x^2y + 3y^2$ $\frac{\partial Q'}{\partial x} = 6x^2 + 2x + 4y + 4xy$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Maka PD diatas terbukti Exact, kemudian selesaikan dengan menggunakan PD Exact

f(x,y) = $\int P'$ dx + C(y)

$= \int(4x^3 + 6x^2y + 2xy + 2xy^2 + 2y^2)dx + C(y)$

$= x^4 + 2x^3y + x^2y + x^2y^2 + 2xy^2 + C(y)$

$\frac{\partial f}{\partial y} = Q'$

$\frac{\partial f}{\partial y} = 2x^3 + x^2 + 2x^2y + 4xy + C'(y) \equiv f(x,y) = 2x^3 + x^2 + 4xy + 2x^2y + 3y^2$

Diperoleh lah $C'(y) = 3y^2 \quad \rightarrow C(y) = y^3$

Dengan demikian penyelesaian umum PD nya adalah : f(x,y) = 0

$x^4 + 2x^3y + x^2y + x^2y^2 + 2xy^2 + y^3 = C$

6. Tentukan solusi umum PD berikut, dengan terlebih dahulu menentukan faktor integrasinya $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$

Penyelesaian :

$(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0 \qquad \ldots\ldots\ldots\ldots (1)$

$P(x,y) = y^3 - 2x^2y \qquad \frac{\partial P}{\partial y} = 3y^2 - 2x^2$

$Q(x,y) = 2xy^2 - x^3 \qquad \frac{\partial Q}{\partial x} = 2y^2 - 3x^2$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 3y^2 - 2x^2 - 2y^2 - 3x^2 = y^2 + x^2$

Kemudian faktor integrasinya dapat di tentukan dari

$u = e^{\int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{yQ - xP} dz}$

$u = e^{\int \frac{x^2+y^2}{y(2xy^2 - x^3) - x(y^3 - 2x^2y)} dz}$

$u = e^{\int \frac{x^2+y^2}{(2xy^3 - x^3y) - (xy^3 - 2x^3y)} dz}$

$u = e^{\int \frac{1}{xy} dz}$

$u = xy$

Substitusi u kepersamaan 1 dalam bentuk u P (x,y)dx + u Q (x,y)dy = 0

$xy(y^3 - 2x^2y)dx + xy(2xy^2 - x^3)dy = 0$

$f(x,y) = \int(xy^4 - 2x^3y^2)dx + C(y)$

$= \frac{1}{2} x^2y^4 - \frac{1}{2} x^4y^2 + C(y)$

$\frac{\partial f}{\partial y} = 2x^2y^3 - x^4y + C'(y) = 2x^2y^3 - x^4y$

$$C'(y) = 0 \quad \rightarrow C(y) = C'$$

$f(x,y) = \frac{1}{2} x^2y^4 - \frac{1}{2} x^4y^2 = -C1 \equiv x^2y^4 - x^4y^2 = C$

Dengan demikian penyelesaian umum PD nya adalah :

$f(x,y) = \frac{1}{2} x^2y^4 - \frac{1}{2} x^4y^2 = 0 \quad \equiv x^2y^4 - x^4y^2 = C$

Latihan soal:

Tentukan Faktor Integrasi dan penyelesaian PD dibawah ini :

1. $(4xy + 3y^2 - x)\,dx + x(x + 2y)\,dy = 0$
2. $(2x^3y + y^4)dx + (2xy^3 + x^4)$
3. $(5x^2 + 4xy + 4x + y^2)dx + (x^2 + 4xy + 5y^2 + 4y)dy = 0$
4. $(x^2 + 3x + 2)dx + (x^2 + x + 1)dy = 0$
5. $(x + x^2y^4) = -y\frac{dx}{dy}$
6. $(10x^2 + 4xy + 2y^2) = (-x^2 - 8xy - 5y^2)\frac{dy}{dx}$
7. $(1 - x^2 + 2y)dx \;+\; x\;dy \;= 0$
8. $(x^3 + y)dx + x(xy - 1)dy = 0$
9. $(5x^2y + 4x + y^3)\,dx + (5xy^2 + 4y + x^3)dy = 0$
10. $(3xy + y^2)dx + (3xy + x^2)dy + 0$

Persamaan Differensial Linier

Bentuk Umum I :

$$\mathbf{y' + Py = Q} \quad \text{………….}(1)$$

Dimana P dan Q adalah fungsi fungsi dari x

Cara pemecahan :

1. Cara Bernaulli

   Misalkan y = uv dimana u dan v adalah fungsi dari x

   $y' = u'.v + u.v'$

   maka persamaan (1) menjadi :

   $y' + Py = Q$

   $u'.v + u.v' + P(uv) = Q$

   $v(u' + P.u) + u.v' = Q$

   ambil syarat ($u' + P.u$) = 0 atau $u.v' = Q$ ……………. (2)

   maka : $\frac{u'}{u}$ = -P atau $\frac{\frac{du}{dx}}{u}$ = - P

   $\frac{du}{u}$ = -P dx ;

   $\int\frac{du}{u} = -\int P\,dx$

   ln u = - ∫ P dx

   ln u = ln $e^{-\int P\,dx}$

   u = $e^{-\int P\,dx}$

   $uv' = Q$

   $e^{-\int P\,dx}.v' = Q$

   $v' = Q.\,e^{\int P\,dx}$

$v = \int e^{\int P\,dx}.Q\,dx + C$

y = uv sehingga diperoleh

$$y = e^{-\int P\,dx} . \left[\int e^{\int P\,dx} . Q\,dx + C \right]$$

2. Cara lagrange merubah konstanta

Ambil $\frac{dy}{dx} + Py = 0$

$dy = -P\,y\,dx$ atau

$$\frac{dy}{y} = -P\,dx$$

$$\int \frac{dy}{y} = \int -P dx$$

$\ln y = -\int P\,dx + C_1 \rightarrow$ misalkan $C_1 = \ln C$

$\ln y = \ln e^{-\int P\,dx} + \ln C$

$y = C\,e^{-\int P\,dx}$ (pandang C sebagai fungsi dari x)

$y = C(x)\,e^{-\int P\,dx}$ …….. (2)

maka $\ln y = -\int P\,dx + \ln C(x)$

differensialkan ke x $\frac{1}{y}\frac{dy}{dx} = -P + \frac{1}{C(x)} . \frac{d\,C(x)}{dx}$

$$\frac{dy}{dx} = \frac{y}{C(x)} . \frac{dC(x)}{dx} - Py$$

$$\frac{dy}{y} + Py = \frac{y}{C(x)} . \frac{dC(x)}{dx} \equiv Q$$

$$\frac{y}{C(x)} . \frac{dC(x)}{dx} \equiv Q$$

$$\frac{C(x).e^{-\int P dx}}{C(x)} . \frac{dC(x)}{dx} = Q$$

$$e^{-\int P\,dx} . \frac{dC(x)}{dx} = Q$$

$$\frac{dC(x)}{dx} = e^{\int P\,dx}.Q$$

$$C(x) = \int e^{\int P\,dx}.Q\,dx + D$$

Maka $y = C(x)\,e^{-\int P\,dx}$

Bentuk Umum II

$$y = e^{-\int P\,dx} . \left[\int e^{\int P\,dx} .Q\,dx + D \right]$$

$\boldsymbol{x' + Px} = \mathbf{Q}$ …………..(3)

Dimana P dan Q adalah fungsi fungsi dari y

Cara pemecahan :

1. Cara Bernaulli

   Misalkan x = uv dimana u dan v adalah fungsi dari y

   $x' = u'.v + u.v'$

   maka persamaan(3) menjadi :

   $x' + Px = Q$

   $u'.v + u.v' + P(uv) = Q$

   $v(u' + P.u) + u.v' = Q$

   ambil syarat $(u' + P.u) = 0$ atau $u.v' = Q$ …………….. (4)

   maka : $\frac{u'}{u} = -P$ atau $\frac{\frac{du}{dy}}{u} = -P$

   $\frac{du}{u} = -P\, dy$;

   $\int \frac{du}{u} = -\int P\, dy$

   $\ln u = -\int P\, dy$

   $\ln u = \ln e^{-\int P\, dy}$

   $u = e^{-\int P\, dy}$

   $uv' = Q$

   $e^{-\int P\, dy}.v' = Q$

   $v' = Q.\, e^{\int P\, dy}$

   $v = \int e^{\int P\, dy}.Q\, dy + C$

   $x = uv$

   $$x = e^{-\int P\, dy} . [\int e^{\int P\, dy} . Q\, dy + C]$$

2. Cara lagrange merubah konstanta

   Ambil $\frac{dx}{dy} + Px = 0$

   $dx = -P\, x\, dy$ atau

   $\frac{dx}{x} = -P\, dy$

   $\int \frac{dx}{x} = \int -P dy$

   $\ln x = -\int P\, dy + C_1 \rightarrow$ misalkan $C_1 = \ln C$

   $\ln x = \ln e^{-\int P\, dy} + \ln C$

   $\ln x = \ln e^{-\int P\, dy}.C$

   $x = C\, e^{-\int P\, dy}$ (pandang C sebagai fungsi dari y)

$x = C(y)\, e^{-\int P\, dy}$ .... (2)

maka $\ln x = -\int P\, dy + \ln C(y)$

diff. ke y $\frac{1}{x}\frac{dx}{dy} = -P + \frac{1}{C(y)} \cdot \frac{d\,C(y)}{dy}$

$\frac{dx}{dy} = \frac{x}{C(y)} \cdot \frac{dC(y)}{dy} - Px$

$\frac{dx}{dy} + Px = \frac{x}{C(y)} \cdot \frac{dC(y)}{dy} \equiv Q$

$\frac{x}{C(y)} \cdot \frac{dC(y)}{dy} \equiv Q$

$\frac{C(y).e^{-\int P dy}}{C(y)} \cdot \frac{dC(y)}{dy} = Q$

$e^{-\int P\, dy} \cdot \frac{dC(y)}{dy} = Q$

$\frac{dC(y)}{dy} = e^{\int P\, dy}.Q$

$C(y) = \int e^{\int P\, dy}.Q\, dy + D$

Maka $x = C(y)\, e^{-\int P\, dy}$

$$x = e^{-\int P\, dy} \cdot \left[\int e^{\int P\, dy}.Q\, dy + D\right]$$

Contoh Soal :

1. Carilah solusi dari soal berikut : $\frac{dy}{dx} - \frac{y}{x} = x$

Penyelesaian :

a. Cara bernaulli

$\frac{dy}{dx} - \frac{y}{x} = x$ $\left(P = \frac{-1}{x},\ Q = x\right)$

Rumus : $y = e^{-\int P\, dx} \cdot \left[\int e^{\int P\, dx} \cdot Q\, dx + C\right]$

$y = e^{-\int -1/x\, dx} \cdot \left[\int e^{\int 1/x\, dx} .x\, dx + C\right]$

$y = e^{\int 1/x\, dx} \cdot \left[\int e^{-\int 1/x\, dx} .x\, dx + C\right]$

$y = x\left[\int \frac{1}{x}.x\, dx + C\right]$

$y = x(x+C)$

$y = x^2 + Cx$

b. Cara lagrange :

$\frac{dy}{dx} - \frac{y}{x} = x$ .... (1)

Ambil $\frac{dy}{dx} - \frac{y}{x} = 0$

$\frac{dy}{dx} = \frac{y}{x}$

$\frac{dy}{y} = \frac{dx}{x}$

ln y = ln x + ln C

ln y = ln x.C

y = Cx atau

y = C(x) .x ………(2) pandang C sebagai fungsi dari x

maka : ln y = ln C(x) + ln x

differensialkan ke x : $\frac{1}{y}\frac{dy}{dx} = \frac{1}{c(x)}\frac{dC(x)}{dx} + \frac{1}{x}$

$\frac{dy}{dx} = \frac{y}{C(x)}\frac{dC(x)}{dx} + \frac{y}{x}$

$\frac{dy}{dx} - \frac{y}{x} = \frac{y}{Cx}.\frac{dC(x)}{dx} \equiv Q$

$\frac{y}{Cx}.\frac{dC(x)}{dx} \equiv Q$

$\frac{y}{Cx}.\frac{dC(x)}{dx} = Q$

$\frac{C(x)x}{Cx}.\frac{dC(x)}{dx} = x$

$x\frac{dC(x)}{dx} = x$

C(x) = dx

C(x) = x + C

y = C(x) .x d

y = (x+C).x

y = x^2 + Cx

y = x^2 + Cx

2. Carilah solusi dari pesamaan berikut $\frac{dx}{dy} - 2xy = 6ye^{y^2}$

Penyelesaian:

Cara Bernauli

$\frac{dx}{dy} - 2xy = 6ye^{y^2}$ → P = -2y Q = $6ye^{y^2}$

$x = e^{-\int P\,dy}.\,[\int e^{\int P\,dy}.\,Q\,dy + C\,]$

$x = e^{-\int -2y\,dy}\left[\int e^{\int -2y\,dy}.6ye^{y^2}dy + C\right]$

$x = e^{y^2}\left[\int e^{-y^2}.6ye^{y^2}dy + C\right]$

$x = e^{y^2}[\int 6y\,dy + C]$

$x = e^{y^2}[3y^2 + C]$

$x = 3y^2e^{y^2} + Ce^{y^2}$

cara lagrange:

$\frac{dx}{dy} - 2xy = 6ye^{y^2}$

Misalkan Q = 0

$\frac{dx}{dy}$ – 2xy = 0

$\frac{dx}{dy}$ = 2xy

$\frac{dx}{x}$ = 2y dy

ln x = y^2+ C_1 → misalkan C_1 = ln C

ln x = ln e^{y^2}+ ln C

ln x = ln e^{y^2}. C

x = e^{y^2}. C → pandang C sebagai fungsi y

x = e^{y^2}. C(y)

ln x = y^2+ C(y)

diff. ke y

$\frac{1}{x}\frac{dx}{dy}$ = 2y + $\frac{1}{C(y)}.\frac{dC(y)}{dy}$

$\frac{dx}{dy}$ = 2xy + $\frac{x}{C(y)}.\frac{dC(y)}{dy}$

$\frac{dx}{dy}$ - 2xy = $\frac{x}{C(y)}.\frac{dC(y)}{dy} \equiv Q$

$\frac{x}{C(y)}.\frac{dC(y)}{dy}$ ≡ Q

$\frac{x}{C(y)}.\frac{dC(y)}{dy}$ = Q

$\frac{e^{y^2}.C(y)}{C(y)}.\frac{dC(y)}{dy}$ = 6ye^{y^2}

$e^{y^2}.\frac{dC(y)}{dy}$ = 6ye^{y^2}

$\frac{dC(y)}{dy}$ = 6y

dC(y) = 6y dy

C(y) = 3y^2+ C

x = e^{y^2}. C(y)

x = e^{y^2}(3y^2+ C)

x = 3y^2 e^{y^2}+ Ce^{y^2}

3. Carilah solusi dari persamaan berikut $(sin^2x - y)\ dx - tg\ x\ dy = 0$

Penyelesaian:

Cara Bernauli

$(sin^2x - y)\ dx - tg\ x\ dy = 0$

Dapat ditulis $(sin^2x - y)\ dx = tg\ x\ dy$

$\frac{dy}{dx} = \frac{sin^2x - y}{tg\ x} = \frac{sin^2}{tg\ x} - \frac{y}{tg\ x}$

$\frac{dy}{dx} + \frac{y}{tg\ x} = \frac{sin^2}{tg\ x}$

$\frac{dy}{dx} + \frac{y}{tg\ x} = sin\ x\ cos\ x$

$(P = \frac{1}{tg\ x}, Q = sin\ x\ cos\ x)$

y = $e^{-\int P\,dx}$. [$\int e^{\int P\,dx}$. Q dx + C]

y = $e^{-\int\frac{1}{tg\,x}dx}.\left[\int e^{\int\frac{1}{tg\,x}}\sin x\cos x\;dx+C\right]$

y = $e^{-\ln\sin x}.[\int e^{\ln\sin x}.\sin x\cos x\;dx+C]$

y = $e^{\ln\sin x^{-1}}[\int e^{\ln\sin x}.\sin x\cos x\;dx+C]$

y = $\frac{1}{\sin x}[\int sin^2x\cos x\;dx+C]$

y = $\frac{1}{\sin x}\left[\frac{1}{3}\;sin^3x+\;C\right]$

y = $\frac{1}{3}\,sin^2x+\;\frac{C}{\sin x}$

cara lagrange

$(sin^2x - y)\ dx - tg\ x\ dy = 0$

Jawab :

$(sin^2x - y)\ dx - tg\ x\ dy = 0$ ……………… *(1)*

(sin x – y) dx = tg x dy

$\frac{dy}{dx} = \frac{sin^2x - y}{tg\ x} = \frac{sin^2}{tg\ x} - \frac{y}{tg\ x}$

atau $\frac{dy}{dx} + \frac{y}{tg\ x} = sin\ x\ cos\ x$

ambil : $\frac{dy}{dx} + \frac{y}{tg\ x} = 0 \quad \rightarrow \frac{dy}{dx} = -\frac{y}{tg\ x}$

$\rightarrow \frac{dy}{y} = -\frac{1}{tg\ x}dx$

Maka ln y = – ln sin x + C_1 ; misalkan C_1 = ln C

ln y = – ln sin x + ln C

ln y = ln C – ln sin x

ln y = ln $\frac{C}{\sin x}$

y = $\frac{C}{\sin x}$ Pandang C sebagai fungsi dari *x*

y = $\frac{C\ (x)}{\sin x}$ …….(2)

ln y = ln C (*x*) – ln sin *x*

differensialkan ke x $\frac{1}{y}\frac{dy}{dx} = \frac{1}{C\ (x)} . \frac{d\ C(x)}{dx} - \frac{1}{tg\ x}$

$\frac{dy}{dx} = \frac{y}{C\ (x)} . \frac{d\ C(x)}{dx} - \frac{y}{tg\ x}$

$\frac{dy}{dx} + \frac{y}{tg\ x} = \frac{y}{C(x)} . \frac{d\ (Cx)}{dx}$ $\equiv$ Q

$\frac{y}{C(x)} . \frac{d\ (Cx)}{dx} \equiv$ Q

$\frac{y}{C(x)} . \frac{d\ (Cx)}{dx}$ = Q

$\frac{\frac{C(x)}{\sin x}}{C(x)} \frac{d\, C(x)}{dx}$ = sin x cos x

$\frac{1}{\sin x} \frac{d\, C(x)}{dx}$ = sin x cos x

d C(x) = sin^2x cos x

C(x) = $\frac{1}{3}$ sin^3 x + C

y = $\frac{C\,(x)}{\sin x}$

y = $\frac{\frac{1}{3}sin^3 x + C}{\sin x}$

y = $\frac{1}{3}$ sin^2 x + $\frac{C}{\sin x}$

4. Carilah solusi dari persamaan berikut $\frac{dy}{dx} = \frac{x^2 + 2y}{x}$

Penyelesaian :

Cara bernauli

$\frac{dy}{dx} = \frac{x^2 + 2y}{x} = 0$

$\frac{dy}{dx} - \frac{1}{x}(x^2 + 2y) = 0$

$\frac{dy}{dx} - \frac{x^2}{x} - \frac{2y}{x} = 0$

$$\frac{dy}{dx} - \frac{2y}{x} = \frac{x^2}{x}$$

$$\frac{dy}{dx} - \frac{2y}{x} = x$$

$\frac{dy}{dx} - \frac{2}{x}y = x \quad \rightarrow \quad (\text{P} = -\frac{2}{x}\ ;\ \text{Q} = x\ )$

Maka,

y = $e^{-\int P\,dx}$. [$\int e^{\int P\,dx}$. Q dx + C]

$= e^{\int \frac{2}{x}dx}$. [$\int e^{\int -\frac{2}{x}dx}$. x dx + C]

= $e^{2\ln x}[\int e^{-2\ln x}$.x dx + C]

= $e^{\ln x^2}[\int e^{\ln x^{-2}}$.x dx + C]

= x^2 [$\int$ x^{-2} . x dx + C]

= x^2 [$\int$ x^{-1} dx + C]

= x^2 [$\int \frac{1}{x}$dx + C]

= x^2 (ln x + C)

Cara lagrange

$\frac{dy}{dx} = \frac{x^2 + 2y}{x}$

$\frac{dy}{dx}$ - $\frac{x^2+2y}{x}$ = 0

$$\frac{dy}{dx} - \frac{x^2}{x} - \frac{2y}{x} = 0$$

$\frac{dy}{dx} - \frac{2y}{x} = \frac{x^2}{x}$...........................(1)

Misalkan, Q = 0

$\frac{dy}{dx} - \frac{2y}{x}$ = 0

$$\frac{dy}{dx} = \frac{2y}{x}$$

dy.x = 2y dx

$$\frac{dy}{y} = \frac{2dx}{x}$$

$$\int\frac{dy}{y} = \int\frac{2dx}{x}$$

ln y =2 ln x + C

ln y = ln x^2 + ln C

ln y = ln x^2.C

y = x^2.C → pandang C sebagai fungsi dari x

y = x^2.C(x)

ln y = ln x^2 + ln C(x) → differensialkan ke x

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{C(x)} . \frac{dC(x)}{dx}$$

$$\frac{dy}{dx} = \frac{2y}{x} + \frac{y}{C(x)} . \frac{d\ C\ (x)}{dx}$$

$\frac{dy}{dx} - \frac{2y}{x} = \frac{y}{C(x)} . \frac{d\ C(x)}{dx} \equiv$ Q

$\frac{y}{C(x)} . \frac{d\ (C(x))}{dx} \equiv Q$

$\frac{y}{C(x)} . \frac{d\ C(x)}{dx} = Q$

$\frac{x^2 C(x)}{C(x)}\ \frac{d\ (C(x))}{dx}$ = x

$\frac{x^2 .d\ (C(x))}{dx}$ = x

x^2.dC(x) = x dx

dC(x) = $\frac{1}{x}$ dx

C(x) = ln x + C

y = x^2.C(x)

y = x^2 (ln x + C)

y = x^2 ln x + Cx^2

5. Carilah solusi dari persamaan berikut y' – y = 2e^x

Penyelesaian :

Cara bernauli

$\frac{dy}{dx} - y = 2e^x$ → P = -1 ; Q = 2e^x

Maka,

y = $e^{-\int P\,dx}$. [$\int e^{\int P\,dx}$. Q dx + C]

= $e^{-\int (-1)\,dx}$. [$\int e^{\int (-1)\,dx}$. 2e^x dx + C]

= $e^{\int dx}$. [$\int e^{\int -dx}$. 2e^x dx + C]

= e^x . [$\int e^{-x}$. 2e^x dx + C]

= e^x . [2x + C]

= $2xe^x + Ce^x$

Cara lagrange

$$\frac{dy}{dx} - y = 2e^x$$

Misalkan Q = 0

$$\frac{dy}{dx} - y = 0$$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{y} = dx$$

ln y = x + C

ln y = ln e^x + ln C

y = e^x. C → pandang C sebagai fungsi x

atau y = e^x. C(x)

ln y = ln e^x + ln C → diferensial ke x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{C(x)}\frac{d\,C(x)}{dx} + 1$$

$$\frac{dy}{dx} = \frac{y}{C(x)}\frac{d\,C(x)}{dx} + y$$

$$\frac{dy}{dx} - y = \frac{y}{C(x)}\frac{d\,C(x)}{dx} \equiv Q$$

$$\frac{y}{C(x)}\frac{d\,C(x)}{dx} \equiv Q$$

$$\frac{y}{C(x)}\frac{d\,C(x)}{dx} = Q$$

$$\frac{e^x\,C\,(x)}{C(x)} . \frac{d\,C(x)}{dx} = 2e^x$$

$$e^x . \frac{d\,C(x)}{dx} = 2e^x$$

$$e^x . d\,C(x) = 2e^x dx \; maka \; \int dC(x) = \int 2\,dx$$

$C(x) = 2x + C$

$y = e^x . C(x)$

$y = e^x (2x + C)$

$y = 2xe^x + Ce^x$

6. Carilah solusi dari persamaan berikut: $\frac{dy}{dx} = \frac{4 \ln x - 2x^2 y}{x^3}$

Penyelesaian :

Cara bernauli

$$\frac{dy}{dx} = \frac{4 \ln x - 2x^2 y}{x^3}$$

$$\frac{dy}{dx} = \frac{4 \ln x}{x^3} - \frac{2}{x} y$$

$\frac{dy}{dx} + \frac{2}{x} y = \frac{4 \ln x}{x^3} \rightarrow$ ($P = \frac{2}{x}$; $Q = \frac{4 \ln x}{x^3}$)

Maka,

$y = e^{-\int P\, dx} . [\int e^{\int P\, dx} . Q\, dx + C]$

$y = e^{-\int \frac{2}{x} dx} . \left[\int e^{\int \frac{2}{x} dx} . \frac{4 \ln x}{x^3} dx + c \right]$

$y = e^{-2 \ln x} . [\int e^{2 \ln x} . \frac{4 \ln x}{x^3} dx + C]$

$y = x^{-2} [\int x^2 . \frac{4 \ln x}{x^3} dx + C]$

$y = x^{-2} [\int \frac{4 \ln x}{x} dx + C]$

$y = x^{-2} [4 \int \frac{\ln x}{x} + C]$

$y = x^{-2} [4 . \frac{1}{2} (\ln x)^2 + C$

$$\frac{y}{x^{-2}} = 2 \ln^2 x + C$$

$x^2 y = 2 \ln^2 x + C$

$y = \frac{2 \ln^2 x + C}{x^2}$

Cara lagrange

$$\frac{dy}{dx} = \frac{4 \ln x - 2x^2 y}{x^3}$$

$$\frac{dy}{dx} - \left(\frac{4 \ln x - 2x^2 y}{x^3} \right) = 0$$

$$\frac{dy}{dx} - \frac{4 \ln x}{x^3} + \frac{2x^2 y}{x^3} = 0$$

$$\frac{dy}{dx} + \frac{2x^2 y}{x^3} = \frac{4 \ln x}{x^3}$$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{4\ln x}{x^3}$$

Ambil $\frac{dy}{dx} + \frac{2}{x}$y = 0

$\frac{dy}{dx} + \frac{2}{x}$y = 0

$\frac{dy}{dx} = -\frac{2}{x}$y

$\frac{dy}{y} = -\frac{2}{x}$dx

ln y = -2 ln x + ln C

y = x^{-2} . C → (pandang C sebagai fungsi x)

y = x^{-2} . C(x)

ln y = -2 ln x + ln C → diferensial ke x

$\frac{1}{y}\frac{dy}{dx} = -\frac{2}{x} + \frac{1}{C(x)}\frac{d\,C(x)}{dx}$

$\frac{dy}{dx} = -\frac{2y}{x} + \frac{y}{C(x)}\frac{d\,C(x)}{dx}$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{y}{C(x)}\frac{d\,C(x)}{dx}$

$\frac{y}{C(x)}\frac{d\,C(x)}{dx} \equiv Q$

$\frac{x^{-2}.C(x)}{C(x)}\frac{d\,C(x)}{dx} = \frac{4\ln x}{x^3}$

$\frac{x^{-2}d\,C(x)}{dx} = \frac{4\ln x}{x^3}$ $maka \int \frac{x^{-2}d\,C(x)}{dx} = \int \frac{4\ln x}{x^3}$

$\int x^3x^{-2}dC(x) = \int 4\ln x\,dx$

$\int xdC(x) = \int 4\ln x\,dx$

$\int dC(x) = \int \frac{4\,ln\,x}{x}\,dx$

$C(x) = 4\int \frac{ln\,x}{x}dx$

$C(x) = 4\left(\frac{1}{2}ln^2x + C\right)$

$C(x) = 2\,ln^2x + C$

$y = x^{-2}.C(x)$

$y = x^{-2}(2\,ln^2x + C)$

$yx^2 = 2\,ln^2x + C$

$y = \frac{2\,ln^2x+C}{x^2}$

Latihan Soal:

Carilah jawab umum dari Persamaan Differensial Linear berikut!

1. $y' - \frac{2y}{x} = x + 1$
2. $\frac{dy}{dx} = \frac{3-xy}{2x^2}$
3. $x^2\,ln\,x\,\frac{dy}{dx} + xy = 1$

4. $\frac{dy}{dx} = \frac{\sin x-(x-y)\cos x}{\sin x}$
5. $dy + (y - 2\sin x)\cos x\, dx = 0$
6. $\frac{dy}{dx} + y = e^x$
7. $(x^2 + 1)\frac{dy}{dx} + xy = x^2$
8. $y' - 2y = \cos 2x$
9. $\frac{dy}{dx} = y \sin x$
10. $(\sin^2 x - y)dx - tg\, x\, dy = 0$
11.

Persamaan Bernauli

Bentuk umum:

y'+Py=Qy^n Atau x'+Px=Qx^n

1. **Persamaan Terhadap Fungsi y**

y'+ Py = Qy^n ....(1)

P dan Q adalah fungsi-fungsi dari x,n≠0,n≠1.

Untuk menyelesaikan persamaan bernauli ini dapat melalui 2 cara yaitu:

a. Persamaan (1) dibagi dengan y^n

Maka $\frac{y'}{y^n} + \frac{P}{y^{n-1}} = Q$ ....(2)

$$\frac{y'}{y^n} + Py^{1-n} = Q$$

Misalkan z =y^{1-n}

Maka: $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{1-n}.y^n\frac{dz}{dx}$$ ....(3)

Kita substitusi pers (3) ke pers (2) menjadi:

$$\frac{1}{1-n}.\frac{y^n}{y^n}\frac{dz}{dx} + \frac{P}{y^{n-1}} = Q$$

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

Ini adalah persamaan differensial linier dan selanjutnya dapat diselesaikan dengan cara bernaulli dan lagrange.

b. $y' + Py = Qy^n$ ....(1)

y=u.v (u,v masing-masing fungsi dari x)

$y' = u'v + uv'$ (turunan aturan perkalian) subsitusi ke pers (1)
$u'v + uv' + P.u.v = Qu^n v^n$
$u(v' + P.v) + u'v = Qu^n v^n$ ....(2)
Kita ambil: $v' + Pv = 0$

$$v' = -Pv$$

$$\frac{v'}{v} = -P \quad \text{atau} \quad \frac{\frac{dv}{dx}}{v} = P$$

$$\int \frac{dv}{v} = \int -Pdx$$

$$\ln v = \int -Pdx$$

$$e^{\ln v} = e^{-\int Pdx}$$

$$v = e^{-\int Pdx}$$

Sehingga pers (2) menjadi:

$$u'v = Qu^n v^n$$

$$\frac{du}{dx} v = Qu^n v^n$$

$$\frac{du}{dx}\frac{1}{u^n} = \frac{Qv^n}{v}$$

$$\frac{du}{u^n} = Qv^{n-1}dx$$

$$\int \frac{1}{u^n} du = \int Qv^{n-1}dx$$

$$\frac{1}{(1-n)} u^{1-n} = \int e^{-\int Pdx^{(n-1)}} Q\, dx + c$$

$$\frac{1}{(1-n)} u^{1-n} = \int e^{(1-n)\int Pdx} Q\, dx + c$$

$$u^{1-n} = \int (1-n) e^{(n-1)\int Pdx}\ Qdx + c$$

Karena y = u.v
Maka $y^{1-n} = u^{1-n}.v^{1-n}$

$$\boldsymbol{y^{1-n} = (1-n).e^{(n-1)\int Pdx}.[\int e^{(1-n)\int Pdx}.Qdx + c]}$$

2. **Persamaan Terhadap Fungsi x**
   $x' + Px = Qx^n$..(1)
   P dan Q adalah fungsi-fungsi dari x, n ≠ 0, n ≠ 1.
   Untuk menyelesaikan persamaan bernaulli ini dapat melalui 2 cara yaitu:

a. Persamaan (1) dibagi dengan X^n
   Maka $\frac{x'}{x^n} + \frac{P}{x^{n-1}}$ = Q ....(2)

Misalkan z =x^{1-n}

Maka: $\frac{dz}{dx} = (1-n)x^{-n}\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{1}{1-n}.x^{n}\frac{dz}{dx}$ ....(3)

Kita substitusi pers(3) ke pers(2) menjadi:

$\frac{1}{1-n}.\frac{x^n}{x^n}\frac{dz}{dx} + \frac{P}{x^{n-1}}$ =Q

$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$

Ini adalah persamaan differensial linier dan selanjutnya dapat diselesaikan dengan cara bernaulli dan lagrange.

b. $x' + Px = Qx^n$ ....(1)

Misalkan x = u.v (u,v masing-masing fungsi dari y)

$x' = u'v + uv'$ (turunan aturan perkalian) subsitusi ke pers (1)

$u'v + uv' + P.u.v = Qu^n v^n$

$u(v' + P.v) + u'v = Qu^n v^n$ ....(2)

Kita ambil: $v' + Pv = 0$

$v' = -Pv$

$\frac{v'}{v} = -P$ atau $\frac{\frac{dv}{dy}}{v} = P$

$\int \frac{dv}{v} = \int -Pdy$

$\ln v = \int -Pdy$

$se^{\ln v} = e^{-\int Pdy}$

$v = e^{-\int Pdy}$ ....(3)

Sehingga pers (2) menjadi:

$u'v = Qu^n v^n$

$\frac{du}{dy}v = Qu^n v^n$

$\frac{du}{dy}\frac{1}{u^n} = \frac{Qv^n}{v}$

$\frac{du}{u^n} = Qv^{n-1}dy$

$\int \frac{1}{u^n}du = \int Qv^{n-1}dy$

$\frac{1}{(1-n)}u^{1-n} = \int e^{-\int Pdy^{(n-1)}}\, Q\, dy + c$

$\frac{1}{(1-n)}u^{1-n} = \int e^{(1-n)\int Pdy}\, Q\, dy + c$

$$u^{1-n} = \int (1-n) e^{(n-1)\int Pdy}\ Qdy + c$$

Karena x=u.v

Maka $x^{1-n} = u^{1-n}.v^{1-n}$

$$\mathbf{x^{1-n} = (1-n).e^{(n-1)\int Pdy}.[\int e^{(1-n)\int Pdy}.Qdy + c]}$$

Contoh Soal:

1. Carilah solusi dari x dy + y dx = x y^2 dx

Penyelesaian :

Cara I :

x dy + y dx = x y^2 dx

: x dx

$\frac{dy}{dx} + \frac{y}{x} = y^2$

$\frac{1}{y^2} . \frac{dy}{dx} + \frac{1}{xy} = 1$ .... (1)

$\frac{y'}{y^2} + \frac{y^{-1}}{x} = 1$

Misalkan : z = y^{-1} ⟶ $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

$\frac{dy}{dx} = -y^2 \frac{dz}{dx}$ .... (2)

Subsitusikan persamaan (2) ke persamaan (1)

$\frac{-y^2}{y^2}\ \frac{dz}{dx} + \frac{1}{xy} = 1$

$-\frac{dz}{dx} + \frac{z}{x} = 1$

$\frac{dz}{dx} - \frac{z}{x} = -1$ (Persamaan Differensial Linear)

Didapat : P = $-\frac{1}{x}$; Q = -1

Maka,

z = $e^{-\int P\,dx}$ [$\int e^{\int P\,dx}$. Q dx + C_1]

z = $e^{\int \frac{1}{x} dx}$ [$\int e^{-\int \frac{1}{x} dx}$. (-1) dx + C_1]

z = $e^{\ln x}$ [$\int e^{-\ln x}$. (-1) dx + C_1]

z = x [$-\int \frac{1}{x}$ dx + C_1]

z = x [- ln x + C_1]

$\frac{1}{y}$ = x [- ln x + C_1] misalkan C_1=ln C

1 = xy [- ln x + ln C]

1 = xylnC/x

1 = xy ln C/x

Cara II :

$\frac{dy}{dx} + \frac{y}{x} = y^2$

Maka didapat, $M = \frac{1}{X}$; N = 1 ; n = 2

$y^{(1-n)} = (1 - n)\, e^{(n-1)\int M\,dx} [\int e^{(1-n)\int M\,dx} . N\,dx + C_1]$

$y^{(1-2)} = (1 - 2)\, e^{(2-1)\int \frac{1}{X}dx} [\int e^{(1-2)\int \frac{1}{X}dx} . 1\,dx + C_1]$

$y^{-1} = - e^{\int \frac{1}{X}dx} [\int e^{-\int \frac{1}{X}dx} . 1\,dx + C_1]$

$y^{-1} = - e^{\ln x} [\int e^{-\ln x} dx + C_1]$

$y^{-1} = - x [\int \frac{1}{x} dx + C_1]$

$\frac{1}{y} = - x [\ln x + C_1]$

$1 = xy [- \ln x - C_1]$ misalkan, $-C_1 = \ln C$

$1 = xy [- \ln x + \ln C]$

$1 = xy \ln C/x$

2. Carilah solusi dari $\frac{dy}{dx} + \frac{y}{x+1} = (2x^2 + 2x + 1)\, y^2$

Penyelesaian:

Cara I :

$\frac{dy}{dx} + \frac{y}{x+1} = (2x^2 + 2x + 1)\, y^2$ .... (1)

$: y^2$

$\frac{\frac{dy}{dx}}{y^2} + \frac{1}{y(x+1)} = (2x^2 + 2x + 1)$ .... (2)

Misalkan $z = \frac{1}{y} \longrightarrow \frac{dz}{dx} = - \frac{1}{y^2} . \frac{dy}{dx}$

$\frac{dy}{dx} = - y^2 \frac{dz}{dx}$ .... (3)

Kita subsitusikan persamaan (3) kpersamaan (2)

$\frac{-y2\,\frac{dz}{dx}}{y^2} + \frac{1}{y(x+1)} = (2x^2 + 2x + 1)$

$-\frac{dz}{dx} + \frac{z}{(x+1)} = (2x^2 + 2x + 1)$

$\frac{dz}{dx} - \frac{z}{(x+1)} = - (2x^2 + 2x + 1)$ (Persamaan Differensial Linier)

didapat, $P = - \frac{1}{(x+1)}$; $Q = -2x^2 - 2x - 1$

Maka,

$z = e^{-\int P\,dx} [\int e^{\int P\,dx} . Q\,dx + C_1]$

$z = e^{\int \frac{1}{(x+1)}dx} [\int e^{-\int \frac{1}{(x+1)}dx} . (-2x^2 - 2x - 1)\,dx + C_1]$

$z = e^{\ln x+1} [\int e^{-\ln x+1} . (-2x^2 - 2x - 1)\,dx + C_1]$

$z = x + 1\ [\int -2x - \frac{1}{(x+1)}\ dx + C_1\ ]$

$\frac{1}{y} = x + 1\ [\ -x^2 - \ln (x + 1) + C_1\ ]$

$1 = y\ (x + 1)\ [\ -x^2 - \ln (x + 1) + C_1\ ]$

$1 = -\ y\ (x + 1)\ [\ x^2 + \ln (x + 1) - C_1\ ]$

$1 + (xy + y)\ [\ x^2 + \ln (x + 1) - C_1\ ] = 0$ misalkan, $-C_1 = C$

$(xy + y)\ [\ x^2 + \ln (x + 1) + C\ ] + 1 = 0$

Cara II :

$\frac{dy}{dx} + \frac{y}{x+1} = (2x^2 + 2x + 1)\ y^2$

Didapat, $M = \frac{1}{x+1}$; $N = 2x^2 + 2x + 1$; $n = 2$

$y^{(1-n)} = (1 - n)\ e^{(n-1)\int M\, dx}\ [\int e^{(1-n)\int M\, dx}\ .\ N\ dx + C_1\ ]$

$y^{(1-2)} = (1 - 2)\ e^{(2-1)\int \frac{1}{x+1} dx}\ [\int e^{(1-2)\int \frac{1}{x+1} dx}\ .\ (2x^2 + 2x + 1)\ dx + C_1\ ]$

$y^{-1} = -\ e^{\ln x+1}\ [\int e^{-\ \ln x+1}\ .\ (2x^2 + 2x + 1)\ dx + C_1\ ]$

$y^{-1} = -\ (x + 1)\ [\int \frac{1}{x+1}\ .\ (2x^2 + 2x + 1)\ dx + C_1\ ]$

$\frac{1}{y} = -\ (x + 1)\ [\int 2x + \frac{1}{x+1}\ dx + C_1\ ]$

$\frac{1}{y} = -\ (x + 1)\ [\ x^2 + \ln x + 1 + C_1\ ]$

$1 = -y\ (x + 1)\ [\ x^2 + \ln x + 1 + C_1\ ]$

$1 + y\ (x + 1)\ [\ x^2 + \ln x + 1 + C_1\ ] = 0$ misal $C_1 = C$

$1 + (xy + y)\ [\ x^2 + \ln x + 1 + C\ ] = 0$

3. Carilah solusi dari $x\frac{dy}{dx} + y = x^3\ y^2 \ln x$

Penyelesaian:

Cara I:

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x^2 \ln x$

Misal :

$z = \frac{1}{y} \longrightarrow \frac{dz}{dx} = -\ \frac{1}{y^2}\frac{dy}{dx}$

$\frac{dy}{dx} = -\ y^2\ \frac{dz}{dx}$

$\frac{1}{y^2}(-\ y^2\ \frac{dz}{dx}) + \frac{z}{x} = x^2 \ln x$

$-\frac{dz}{dx} + \frac{z}{x} = x^2 \ln x$

$\frac{dz}{dx} - \frac{z}{x} = -\ x^2 \ln x$ …………(Persamaan Differensial Linear)

Maka :

$P = -\frac{1}{x} \quad Q = -x^2 \ln x$

$Z = e^{-\int P\,dx}\,[\int e^{\int P\,dx} \,.\, Q\,dx + C\,]$

$Z = e^{-\int -\frac{1}{x}dx}\,[\int e^{\int -\frac{1}{x}dx} \,.\, (-x^2 \ln x)\,dx + C\,]$

$Z = e^{\ln x}\,[\int e^{-\ln x} \,.\, (-x^2 \ln x)\,dx + C\,]$

$Z = x\,[\int \frac{1}{x} \,.\, (-x^2 \ln x)\,dx + C\,]$

$Z = x\,[\int -x \ln x)\,dx + C\,]$

$Z = x\,[-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + c_1\,]$

$\frac{1}{y} = -\frac{1}{2}x^3 \ln x + \frac{1}{4}x^3 + c_1 x$

$1 = -\frac{1}{2}x^3 y \ln x + \frac{1}{4}x^3 y + c_1 xy$

$2 = -x^3 y \ln x + \frac{1}{2}x^3 y + 2c_1 xy$

$x^3 y \ln x - \frac{1}{2}x^3 y - 2c_1 xy + 2 = 0$

$xy\,(x^2 \ln x - \frac{1}{2}x^2 - 2c_1) + 2 = 0$

$\longrightarrow$ Misal ($-2c_1 = C$)

$xy\,(x^2 \ln x - \frac{1}{2}x^2 + C) + 2 = 0$

CARA II

$x\frac{dy}{dx} + y = x^3 y^2 \ln x$

$\frac{dy}{dx} + \frac{y}{x} = x^2 \ln x\, y^2$

Maka :

$P = \frac{1}{x} \quad Q = x^2 \ln x \quad n = 2$

$y^{(1-n)} = (1-n)\, e^{(n-1)\int P\,dx}\,[\int e^{(1-n)\int P\,dx} \,.\, Q\,dx + C\,]$

$y^{(1-2)} = (1-2)\, e^{(2-1)\int \frac{1}{x}dx}\,[\int e^{(1-2)\int \frac{1}{x}dx} \,.\, (x^2 \ln x)\,dx + C\,]$

$y^{-1} = -e^{\int \frac{1}{x}dx}\,[\int e^{-\int \frac{1}{x}dx} \,.\, (x^2 \ln x)\,dx + C\,]$

$y^{-1} = -e^{\ln x}\,[\int e^{-\ln x} \,.\, (x^2 \ln x)\,dx + C\,]$

$y^{-1} = -x\,[\int \frac{1}{x} \,.\, (x^2 \ln x)\,dx + C\,]$

$y^{-1} = -x\,[\int x \ln x\,dx + C\,]$

$\int x \ln x \, dx = uv - \int v \, du$ $\longrightarrow$ Misal : $u = \ln x$ $du = \frac{1}{x} dx$

$dv = x \, dx$ $v = \frac{1}{2}x^2$

$\int x \ln x \, dx = \ln x . \frac{1}{2}x^2 - \int \frac{1}{2}x^2 . \frac{1}{x} dx$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$

$y^{-1} = - x [\int x \ln x \, dx + C]$

$\frac{1}{y} = - x [\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C]$

$1 = - \frac{1}{2}x^3 y \ln x + \frac{1}{4}x^3 y - Cxy$

$2 = -x^3 y \ln x + \frac{1}{2}x^3 y - Cxy$

$x^3 y \ln x - \frac{1}{2}x^3 y + Cxy + 2 = 0$

$xy (x^2 \ln x - \frac{1}{2}x^2 + C) + 2 = 0$

4. Carilah solusi dari $x^2 y - x^3 y = y^4 \cos x$

Penyelesaian :

Cara I:

$y' - \frac{y}{x} = -\frac{y^4 \cos x}{x^3}$

$\frac{y'}{y^4} - \frac{1}{xy^3} = -\frac{\cos x}{x^3}$ ....(2)

misalkan $z = \frac{1}{y}$ $\longrightarrow$ $\frac{dz}{dx} = -\frac{3}{y^4}\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{y^4}{3}\frac{dz}{dx}$ ....(3)

.

masukkan (3) ke dalam (2)

$$\frac{\frac{-y^4}{3}\frac{dz}{dx}}{y^4} - \frac{z}{x} = -\frac{\cos x}{x^3}$$

$$-\frac{1}{3}\frac{dz}{dx} - \frac{z}{x} = -\frac{\cos x}{x^3}$$

$\frac{dz}{dx} + \frac{3z}{x} = \frac{3\cos x}{x^3}$ (Adalah Persamaan Diferensial Linear)

$(P = \frac{3}{X} , \ Q = \frac{3COSX}{X^3})$

$z = e^{-\int Pdx} [\int e^{\int Pdx} . Qdx + C]$

$z= e^{-\int \frac{3}{x}dx} [\int e^{\int \frac{3}{x}dx} . \frac{3COSx}{x^3} dx + C]$

$z= e^{-3\ln x} [\int e^{3\ln x} . \frac{3COSx}{x^3} dx + C]$

$z= \frac{1}{x^3} [\int x^3 . \frac{3COSx}{x^3} dx + C]$

$z= \frac{1}{x^3} [3\int \cos x\, dx + C]$

$z= \frac{1}{x^3} (3 \text{ Sinx } + C)$

$\frac{1}{y^3} = \frac{1}{x^3} (3 \text{ Sinx } + C)$

$x^3 = y^3(3 \text{ Sinx } + C)$

Cara II

$(P =-\frac{1}{x} , Q = -\frac{Cosx}{x^3}, n = 4)$

$y^{1-n} = (1 - n). e^{(n-1)\int Pdx}.[\int e^{(1-n)\int Pdx} . Qdx + c]$

$y^{1-4} = (1 - 4). e^{(4-1)\int -\frac{1}{x}dx}.[\int e^{(1-4)\int -\frac{1}{x}dx} . -\frac{Cosx}{x^3} dx + c]$

$y^{-3} = (-3). e^{(3)\int -\frac{1}{x}dx}.[\int e^{(-3)\int -\frac{1}{x}} . -\frac{Cosx}{x^3} dx + c]$

$y^{-3} = (-3). e^{-3\ln x}.[\int e^{3\ln x} . -\frac{Cosx}{x^3} dx + c]$

$y^{-3} = (-3). e^{-3\ln x}.[\int e^{3\ln x} . -\frac{Cosx}{x^3} dx + c]$

$y^{-3} = (-3). \frac{1}{x^3} [\int x^3 . -\frac{Cosx}{x^3} dx + c]$

$y^{-3} = (-3). \frac{1}{x^3} [-\int \cos x\, dx + c]$

$y^{-3} = -\frac{3}{x^3} (-\sin x + c)$

$\frac{1}{y^3} = -\frac{3}{x^3} (-\sin x + c)$

$x^3 = y^3(3 \sin x + c)$

5. carilah solusi dari $\frac{dy}{dx} + y \cot x = y^3 \sin 2x$

$$y' + y \cot x = y^3 \sin 2x$$

Penyelesaian:

Cara I:

$\frac{y'}{y^3} - \frac{\cot x}{y^2} = \sin 2x$ ....(1)

misalkan $z = \frac{1}{y^2}$ $\longrightarrow$ $\frac{dz}{dx} = -\frac{2}{y^3} \frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{y^3}{2} \frac{dz}{dx}$ ....(2).

masukkan (2) ke dalam (1)

$$\frac{\frac{-y^3}{2}\frac{dz}{dx}}{y^3} - \text{ zcotx} = \text{ sin2x}$$

$$-\frac{1}{2}\ \frac{dz}{dx} + \text{ zcotx} = \text{ Sin 2x}$$

$$\frac{dz}{dx} - \text{2zcotx} = -\text{ 2 sin2x} \quad \text{(Adalah PD Linear)}$$

$$(\text{ P} = -2\text{cotx}, \ Q = -2\text{sin2x})$$

$$z = e^{-\int Pdx}\ [\int e^{\int Pdx} . Qdx + C\ ]$$

$$z = e^{-\int -2cotxdx}\ [\int e^{-\int 2cotxdx} . -2\sin 2x\ dx + C\ ]$$

$$z = e^{2lnsinx}\ [\int e^{-2\ lnsinx} . -2\sin 2x\ dx + C\ ]$$

$$z = \sin^2 x\ [\int \frac{1}{\sin^2 x} . -2\sin 2x\ dx + C\ ]$$

$$z = \sin^2 x\ [-2\int \frac{1}{\sin^2 x} . \sin 2x\ dx + C\ ]$$

$$z = \sin^2 x\ (-2\int \frac{2Sinx\ .cosx}{\sin^2 x}\ dx + C)$$

$$z = \sin^2 x\ (-2\int 2\frac{cosx}{sinx}\ dx\ + C)$$

$$\frac{1}{y^2} = \sin^2 x(-2(2lnsinx)\ + C)$$

$$\frac{1}{y^2} = \sin^2 x(-4lnsinx)\ + C)$$

$$1 + y^2\sin^2 x(4lnsinx)\ + C) = 0$$

Cara II

$$(P = \text{cotx}\ , Q = \sin 2x, n = 3)$$

$$y^{1-n} = (1-n). e^{(n-1)\int Pdx}.[\int e^{(1-n)\int Pdx} . Qdx + c]$$

$$y^{-2} = (1-3). e^{(3-1)\int cotxdx}.[\int e^{(1-3)\int c\ otxdx} . \sin 2x\ dx + c]$$

$$y^{-2} = (-2). e^{(2)\int cotxdx}.[\int e^{(-2)\int cotxdx} . \sin 2xdx + c]$$

$$y^{-2} = (-2). e^{2lnsinx}.[\int e^{-2\int\ lnsinx} . \sin 2xdx + c]$$

$$y^{-2} = (-2). \sin^2 x.[\int \frac{\sin 2x}{\sin^2 x} dx + c]$$

$$y^{-2} = (-2). \sin^2 x[\ \int \frac{2Sinx\ .cosx}{\sin^2 x} dx + c]$$

$$y^{-2} = (-2)\ \sin^2 x[2\int \frac{cosx}{sinx} dx + c]$$

$$\frac{1}{y^2} = (-2)\ \sin^2 x(2lnsinx + c)$$

$$\frac{1}{y^2} = \sin^2 x(-4lnsinx)\ + C$$

$$1 + y^2\sin^2 x(4lnsinx)\ + C) = 0$$

Latihan Soal!

Carilah solusi dari persamaan differensial Bernauli berikut :

1. $y + 2y' = y^3(x-1)$
2. $y - 2x\frac{dy}{dx} = x(x+1)\,y^3$
3. $y' - y = xy^6$
4. $(x + 2y^3)y' = y$
5. $x\frac{dy}{dx} + y = (x^5 + 2x^4 + 3x^2)y^4$
6. $x\,dy + y\,dx = x\,y^2 dx$
7. $2y - 3\frac{dy}{dx} = y^4 e^{3x}$
8. $\frac{dy}{dx} + y = xy^3$
9. $y + 2y' = y^3\,(x-1)$
10. $x\frac{dy}{dx} + y = xy^2 \ln$

BAB II
PERSAMAAN DIFERENSIAL
LINEAR TINGKAT n

Persamaan Diferensial Linear Homogen Dengan Koefisien Tetap

Bentuk Umum :

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \ldots\ldots\ldots (1)$$

Dimana $a_0, a_1, a_2, \ldots a_n$ bilangan − bilangan tetap ,

dapat diselesaikan dengan substitusi : $y = e^{tx}$

maka: $\frac{dy}{dx} = te^{tx}, \frac{d^2y}{dx^2} = t^2e^{tx}, \frac{d^3y}{dx^3} = t^3e^{tx} \ldots \frac{d^ny}{dx^n} = t^ne^{tx}$

bila dimasukkan ke dalam pers (1)menjadi :

$$a_0t^ne^{tx} + a_1t^{n-1}e^{tx} + a_2t^{n-2}e^{tx} + \ldots + a_{n-1}te^{tx} + a_ne^{tx} = 0$$

$$e^{tx}(a_0t^n + a_1t^{n-1} + a_2t^{n-2} + \ldots + a_{n-1}t + a_n) = 0 \ : \ e^{tx}$$

$(a_0t^n + a_1t^{n-1} + a_2t^{n-2} + \ldots + a_{n-1}t + a_n) =$
0 (pers. karakteristik (1))

Ada 3 ketentuan yang berlaku yaitu :

- bila akar − akar $t_1 \neq t_2 \neq t_3 \neq \ldots\ldots t_n$

  maka jawab umumnya :

  $$\boldsymbol{y = C_1e^{t_1x} + C_2e^{t_2x} + \ldots\ldots + C_ne^{t_nx}}$$

- bila akar − akar kompleks, mis : $t_1 = a + bi, t_2 = a - bi,$

  maka jawab umumnya :

  $$\boldsymbol{y = C_1e^{(a+bi)x} + C_2e^{(a-bi)x}}$$
  $$\boldsymbol{= e^{ax}(C_1e^{bix} + C_2e^{-bix})}$$
  $$\boldsymbol{= e^{ax}(C_1\cos bx + C_2\sin bx)}$$

- bila akar − akarnya sama atau rangkap

  $t_1 = t_2 = t_3 = \ldots\ldots\ldots = t_n = t$

  maka jawab umumnya :

  $$\boldsymbol{y = C_1e^{tx} + C_2xe^{tx} + C_3x^2e^{tx} + \ldots + C_nx^{n-1}e^{tx}}$$
  $$\boldsymbol{= (C_1 + C_2x + C_3x^2 + \ldots + C_nx^{n-1})e^{tx}}$$

Contoh Soal

1. Carilah jawab umum dari $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Penyelesaian:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Misalkan : $y = e^{tx} \rightarrow \frac{dy}{dx} = te^{tx}, \frac{d^2y}{dx^2} = t^2e^{tx}, \frac{d^3y}{dx^3} = t^3e^{tx}$

$$t^3e^{tx} - 3t^2e^{tx} + 3te^{tx} - e^{tx} = 0$$

$e^{tx}(t^3 - 3t^2 + 3t - 1) = 0 \quad : e^{tx}$

$t^3 - 3t^2 + 3t - 1 = 0$ (persamaan karakteristik)

$(t-1)^3 = 0$

$t_1 = 1, t_2 = 1, t_3 = 1$ (sesuai jawab umum 1)

maka :

$y = C_1e^x + C_2xe^x + C_3x^2e^x$

$y = (C_1 + C_2x + C_3x^2)e^x$

2. Carilah jawab umum dari $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Penyelesaian :

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0 \quad \ldots\ldots\ldots (1)$$

$t^2e^{tx} - 2te^{tx} + e^{tx} = 0$

$t^2 - 2t + 1 = 0$ (persamaan karakteristik)

$(t-1)(t-1) = 0$

$t_1 = 1, t_2 = 1$ (sesuai jawab umum 1)

maka :

$y = C_1e^x + C_2xe^x$

$y = (C_1 + C_2x)e^x$

3. Carilah jawab umum dari $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

Penyelesaian:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0 \quad \ldots\ldots\ldots (1)$$

$t^2e^{tx} + 4te^{tx} + 5e^{tx} = 0$

$t^2 + 4t + 5 = 0$ (persamaan karakteristik)

$$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$= \dfrac{-4 \pm \sqrt{16-20}}{2}$

$= \dfrac{-4 \pm \sqrt{-4}}{2}$

$= \dfrac{-4 \pm 2i}{2}$

$= -2 \pm i$

$t_1 = -2 + i$, $t_2 = -2 - i$ (sesuai jawab umum 2)

maka :

$y = e^{-2x} (C_1 \cos x + C_2 \sin x)$

4. Carilah jawab umum dari $\frac{d^5y}{dx^5} + 8\frac{d^3y}{dx^3} + 16\frac{dy}{dx} = 0$

Penyelesaian:

$\frac{d^5y}{dx^5} + 8\frac{d^3y}{dx^3} + 16\frac{dy}{dx} = 0$

$t^5 + 8t^3 + 16t = 0$ (persamaan karakteristik)

$t(t^4 + 8t^2 + 16) = 0$

$t(t^2 + 4)^2 = 0$

$t(t^2 + 4)(t^2 + 4) = 0$

$t_1 = 0, t_2 = 2i , t_3 = -2i , t_4 = 2i , t_5 = -2i$

(sesuai jawab umum 1, 2, dan 3)

maka :

$y = C_1 + C_2 \cos 2x + C_3 \sin 2x + C_4 x \cos 2x + C_5 x \sin 2x$

$y = C_1 + (C_2 + C_4 x) \cos 2x + (C_3 + C_5 x) \sin 2x$

5. Carilah jawab umum dari $\left(\frac{dy}{dx} - y\right)^2 \left(\frac{dy}{dx} + y\right)^3$

Penyelesaian:

$\left(\frac{dy}{dx} - y\right)^2 \left(\frac{dy}{dx} + y\right)^3 = 0$ . . . . (1)

$(t-1)^2 (t+1)^3 = 0$ (pers. karakteristik)

$(t-1)(t-1)(t+1)(t+1)(t+1) = 0$

$t_1 = t_2 = 1$, $\quad t_3 = t_4 = t_5 = -1$

maka :

$y = C_1 e^x + C_2 x e^x + C_3 e^{-x} + C_4 x e^{-x} + C_5 x^2 e^{-x}$

$y = (C_1 + C_2 x) e^x + (C_3 + C_4 x + C_5 x^2) e^{-x}$

6. Carilah jawab umum dari $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

Penyelesaian:

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$
$t^3 - t^2 + t - 1 = 0$
$(t-1)(t^2+1) = 0$
$t_1 = 1, t_2 = i, \quad t_3 = -i$
maka :
$y = C_1 + C_2 \cos x + C_3 \sin x$

7. Carilah jawab umum dari $\left(\frac{d^2y}{dx^2} + 9y\right)^2 \left(\frac{dy}{dx} - 5y\right)^3$

Penyelesaian:
$$\left(\frac{d^2y}{dx^2} + 9y\right)^2 \left(\frac{dy}{dx} - 5y\right)^3 \;.\;.\;.\;.\;.\; (1)$$
$(t^2 + 9)^2 (t-5)^3 = 0$ (pers. karakteristik)
$(t^2 + 9)(t^2 + 9)(t-5)(t-5)(t-5) = 0$

$t_{1,2} = \pm 3i\,,\ t_{3,4} = \pm 3i\,,\ t_5 = t_6 = t_7 = 5$ (sesuai jawab umum 2 dan 3)
maka :
$y = (C_1 + C_3x)^{3ix} + (C_2 + C_4x)^{-3ix} + (C_5 + C_6x + C_7x^2)e^{5x}$
$\mathrm{y} = (C_1 + C_3x)\,cos3x + (\mathrm{C}_2 + C_4x)sin\,3x + (C_5 + C_6x + C_7x^2)e^{5x}$

Latihan Soal!

Hitunglah jawab umum Persamaan Diferensial Homogen berikut:

1. $\frac{d^2y}{dx^2} + 4y = 0$
2. $\frac{d^4y}{dx^4} + 18\frac{d^2y}{dx^2} + 81y = 0$
3. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$
4. $(\frac{d^2y}{dx^2} - y)\,(\frac{dy}{dx} - 5y)^2 = 0$
5. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$
6. $\frac{d^4y}{dx^4} + 18\frac{d^2y}{dx^2} + 81\,y = 0$
7. $\frac{d^2y}{dx^2} - 3i\frac{dy}{dx} - 2y = 0$
8. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$
9. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 14\,\frac{dy}{dx} = 0$
10. $\frac{d^4y}{dx^4} - y = 0$
11. $\frac{d^2y}{dx^2} - 7i\frac{dy}{dx} - 10y = 0$

12. $\left(\frac{dy}{dx} - 2y\right)^6 \left(\frac{dy}{dx} + 3y\right)^4 = 0$

13. $\frac{d^3y}{dx^3} - \frac{d^2x}{dx^2} + 4\frac{dy}{dx} - 4y = 0$

14. $\frac{d^5y}{dx^5} - 6\frac{d^4y}{dx^4} + 12\frac{d^3y}{dx^3} - 8\frac{d^2y}{dx^2} = 0$

15. $\left(\frac{d^2y}{dx^2} + y\right)\left(\frac{d^2y}{dx^2} + \frac{dy}{dx} + y\right)\left(\frac{dy}{dx} + 3y\right) = 0$

Persamaan Diferensial Linear Homogen yang Koefisien – Koefisiennya Fungsi Istimewa dari x

Bentuk Umum :

$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_ny = 0$ …… (1)

Dengan; $a_0 = A_0(a+bx)^n, a_1 = A_1(a+bx)^{n-1}, \dots, a_n = A_n$

misalkan: $a + bx = e^u$ (kedua ruas kita ln kan)

$\ln(a+bx) = lne^u$

$\ln(a+bx) = u$ (turunkan terhadap x)

$\frac{b}{a+bx} = \frac{du}{dx}$

Maka diperoleh: $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ (substitusikan nilai $\frac{du}{dx}$)

$\frac{dy}{dx} = \frac{dy}{du}.\frac{b}{a+bx}$ (kedua ruas dikali$(a+bx)$)

$(a+bx)\frac{dy}{dx} = b\frac{dy}{du}$ ………...(2)

Persamaan (2) didifferensialkan terhadap x.

$(a+bx)\frac{d^2y}{dx^2} + b\frac{dy}{dx} = b\frac{dy}{du}.\frac{dy}{dx}$ (substitusikan nilai $\frac{dy}{dx}$)

$(a+bx)\frac{d^2y}{dx^2} + b\frac{dy}{dx} = b\frac{dy}{du}(\frac{dy}{du}.\frac{b}{a+bx})$

$(a+bx)\frac{d^2y}{dx^2} + b\frac{dy}{dx} = b\frac{d^2y}{du^2}.\frac{b}{a+bx}$ (kedua ruas dikali$(a+bx)$)

$(a+bx)^2\frac{d^2y}{dx^2} + b(a+bx)\frac{dy}{dx} = b^2\frac{d^2y}{du^2}$ (substitusikan nilai $(a+bx)\frac{dy}{dx}$)

$(a+bx)^2\frac{d^2y}{dx^2} + b\left(b\frac{dy}{du}\right) = b^2\frac{d^2y}{du^2}$

$(a+bx)^2\frac{d^2y}{dx^2} + b^2\frac{dy}{du} = b^2\frac{d^2y}{du^2}$

$(a+bx)^2\frac{d^2y}{dx^2} = b^2\frac{d^2y}{du^2} - b^2\frac{dy}{du}$………..(3)

Persamaan (3) didifferensilkan terhadap x.

$$(a+bx)^2\frac{d^3y}{dx^3}+2b(a+bx)\frac{d^2y}{dx^2}=(b^2\frac{d^2y}{du^2}-b^2\frac{dy}{du})\frac{dy}{dx}$$

$$(a+bx)^2\frac{d^3y}{dx^3}+2b(a+bx)\frac{d^2y}{dx^2}=\left(b^2\frac{d^2y}{du^2}-b^2\frac{dy}{du}\right)(\frac{dy}{du}.\frac{b}{a+bx})$$

$$(a+bx)^2\frac{d^3y}{dx^3}+2b(a+bx)\frac{d^2y}{dx^2}=\left(\frac{b^3}{a+bx}\frac{d^3y}{du^3}\right)-(\frac{b^3}{a+bx}\frac{d^2y}{du^2})$$

$$(a+bx)^3\frac{d^3y}{dx^3}+2b(a+bx)^2\frac{d^2y}{dx^2}=b^3\frac{d^3y}{du^3}-b^3\frac{d^2y}{du^2}$$

$$(a+bx)^3\frac{d^3y}{dx^3}+2b(b^2\frac{d^2y}{du^2}-b^2\frac{dy}{du})=b^3\frac{d^3y}{du^3}-b^3\frac{d^2y}{du^2}$$

$$(a+bx)^3\frac{d^3y}{dx^3}+2b^3\frac{d^2y}{du^2}-2b^3\frac{dy}{du}=b^3\frac{d^3y}{du^3}-b^3\frac{d^2y}{du^2}$$

$$(a+bx)^3\frac{d^3y}{dx^3}=b^3\frac{d^3y}{du^3}-b^3\frac{d^2y}{du^2}-2b^3\frac{d^2y}{du^2}+2b^3\frac{dy}{du}$$

$$(a+bx)^3\frac{d^3y}{dx^3}=b^3\frac{d^3y}{du^3}-3b^3\frac{d^2y}{du^2}+2b^3\frac{dy}{du}\text{ ....}(4)$$

Persamaan (4) didifferensialkan terhadap x.

$$(a+bx)^3\frac{d^4y}{dx^4}+3b(a+bx)^2\frac{d^3y}{dx^3}=(b^3\frac{d^3y}{du^3}-3b^3\frac{d^3y}{du^3}+2b^3\frac{dy}{du})\frac{dy}{dx}$$

$$(a+bx)^3\frac{d^4y}{dx^4}+3b(a+bx)^2\frac{d^3y}{dx^3}$$
$$=(b^3\frac{d^3y}{du^3}-3b^3\frac{d^2y}{du^2}+2b^3\frac{dy}{du})(\frac{dy}{du}\frac{b}{(a+bx)})$$

$$(a+bx)^3\frac{d^4y}{dx^4}+3b(a+bx)^2\frac{d^3y}{dx^3}$$
$$=(\frac{b^4}{a+bx}\frac{d^4y}{du^4})-(\frac{3b^4}{a+bx}\frac{d^4y}{du^4})+(\frac{2b^4}{a+bx}\frac{d^2y}{du^2})$$

$$(a+bx)^4\frac{d^4y}{dx^4}+3b(a+bx)^3\frac{d^3y}{dx^3}=b^4\frac{d^4y}{du^4}-3b^4\frac{d^3y}{du^3}+2b^4\frac{d^2y}{du^2}$$

$$(a+bx)^4\frac{d^4y}{dx^4}+3b\left(b^3\frac{d^3y}{du^3}-3b^3\frac{d^2y}{du^2}+2b^3\frac{dy}{du}\right)=b^4\frac{d^4y}{du^4}-3b^4\frac{d^3y}{du^3}+2b^4\frac{d^2y}{du^2}$$

$$(a+bx)^4\frac{d^4y}{dx^4}+3b^4\frac{d^3y}{du^3}-9b^4\frac{d^2y}{du^2}+6b^4\frac{dy}{du}=b^4\frac{d^4y}{du^4}-3b^4\frac{d^3y}{du^3}+2b^4\frac{d^2y}{du^2}$$

$$\boldsymbol{(a+bx)^4\frac{d^4y}{dx^4}=b^4\frac{d^4y}{du^4}-6b^4\frac{d^3y}{du^3}+11b^4\frac{d^2y}{du^2}-6b^4\frac{dy}{du}}\textbf{ ....(5)}$$

Untuk pangkat ke-n yaitu $(a+bx)^n\frac{d^ny}{dx^n}=$ ....(n) kita tidak dapat menentukan rumusnya karena koefisien-koefisien yang terdapat pada persamaan (2),(3),(4) dan (5) tidak memiliki pola yang beraturan. Sehingga untuk menentukan rumus pangkat ke-n kita harus mencarinya secara berurutan. Dengan kata lain, jika kita ingin mencari turunan ke-9 maka kita harus mencari mulai dari turunan pertama sampai turunan yang ke-9 secara berurutan.

Dan jika, persamaan-persamaan tersebut disubtitusikan ke persamaan (1) maka diperolehlah Persamaan Differensial dengan koefisien-koefisien yang tetap.

Contoh soal :

1. Carilah jawab umum dari $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ ………..(1)

Penyelesaian :

Misalkan : $x = e^u$ (kedua ruas kita ln kan)

$\ln x = \ln e^u$

$\ln x = u$ (turunkan terhadap x)

$\frac{1}{x} = \frac{du}{dx}$

Maka diperoleh: $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ (substitusikan nilai $\frac{du}{dx}$)

$\frac{dy}{dx} = \frac{dy}{du}.\frac{1}{x}$ (kedua ruas dikali x)

$x\frac{dy}{dx} = \frac{dy}{du}$ ………………………………..(2)

Persamaan (2) didiferensialkan terhadap x.

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{du}.\frac{dy}{dx}$ (subtitusikan nilai $\frac{dy}{dx}$)

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{du}(\frac{dy}{du}.\frac{1}{x})$

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{du^2}.\frac{1}{x}$ (kedua ruas dikali x)

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{du^2}$ (subtitusikan nilai$x\frac{dy}{dx}$)

$x^2\frac{d^2y}{dx^2} + \frac{dy}{du} = \frac{d^2y}{du^2}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ ……………………..(3)

Substitusikan persamaan (2) dan (3) ke persamaan (1)

$(\frac{d^2y}{du^2} - \frac{dy}{du}) - \frac{dy}{du} + y = 0$

$\frac{d^2y}{du^2} - 2\frac{dy}{du} + y = 0$ (Persamaan differensial dengan koefisien-koefisien tetap)

Persamaan karakteristik : $t^2 - 2t + 1 = 0$

$(t-1)^2 = 0$

$t_1 = 1\ ;\ t_2 = 1$

Jawab umum: $y = (C_1 + C_2u)e^u$

$y = (C_1 + C_2 lnx)e^{lnx}$

$y = (C_1 + C_2 lnx)x$
$y = C_1 x + C_2 xlnx$

2. Carilah jawab umum dari persamaan diferensial $x^2 \frac{d^2y}{dx^2} - 2y = 0$

Penyelesaian:

Misalkan: $x = e^u$ (kedua ruas kita ln kan)

$\ln x = \ln e^u$

$\ln x = u$ (turunkan terhadap x)

$\frac{1}{x} = \frac{du}{dx}$

Maka diperoleh: $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$ (subtitusikan nilai $\frac{du}{dx}$)

$\frac{dy}{dx} = \frac{dy}{du}\frac{1}{x}$ (kedua ruas dikali x)

$x\frac{dy}{dx} = \frac{dy}{du}$……………………………………(1)

Persamaan (2) didifferensialkan terhadap x.

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{du} . \frac{dy}{dx}$ (subtitusikan nilai $\frac{dy}{dx}$)

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{du}(\frac{dy}{du} . \frac{1}{x})$

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{du^2} . \frac{1}{x}$ (kedua ruas dikali x)

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{du^2}$ (subtitusikan nilai $x\frac{dy}{dx}$)

$x^2\frac{d^2y}{dx^2} + \frac{dy}{du} = \frac{d^2y}{du^2}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ ……………………………………………………(2)

Substitusikan persamaan (1) dan (2) ke soal

$\frac{d^2y}{du^2} - \frac{dy}{du} - 2y = 0$ (Persamaan differensial dengan koefisien-koefisien tetap)

Persamaan karakteristik: $t^2 - t - 2 = 0$

$$(t-2)(t+1)=0$$
$$t_1=2, t_2=-1$$

Jawab umum: $y=C_1e^{t_1u}+C_2e^{t_2u}$

$$y=C_1e^{2lnx}+C_2e^{-lnx}$$
$$y=C_1e^{lnx^2}+C_2e^{lnx^{-1}}$$
$$y=C_1x^2+C_2\frac{1}{x}$$

3. Carilah jawab umum dari $x\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$

Penyelesaian:

$x\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$ ....(1)

Misalkan: $x=e^u$ (kedua ruas kita ln kan)

$\ln x=\ln e^u$

$\ln x=u$ (turunkan terhadap x)

$\frac{1}{x}=\frac{du}{dx}$

Maka diperoleh: $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$ (substitusikan nilai $\frac{du}{dx}$)

$\frac{dy}{dx}=\frac{dy}{du}.\frac{1}{x}$ (kedua ruas dikali x)

$x\frac{dy}{dx}=\frac{dy}{du}$ ....(2)

Persamaan (2) didifrensialkan terhadap x.

$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{dy}{du}.\frac{dy}{dx}$ (subtitusikan nilai $\frac{dy}{dx}$)

$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{dy}{du}(\frac{dy}{du}.\frac{1}{x})$

$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d^2y}{du^2}.\frac{1}{x}$ (subtitusikan nilai $\frac{dy}{dx}$)

$$x\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{du}=\frac{1}{x}\frac{d^2y}{du^2}$$

$x\frac{d^2y}{dx^2}=\frac{1}{x}\frac{d^2y}{du^2}-\frac{1}{x}\frac{dy}{du}$ ....(3)

Substitusikan persamaan (2) dan (3) ke persamaan (1)

$$\left(\frac{1}{x}\frac{d^2y}{du^2} - \frac{1}{x}\frac{dy}{du}\right) - \left(\frac{1}{x}\frac{dy}{du}\right) = 0$$

$$\frac{d^2y}{du^2} - \frac{dy}{du} - \frac{dy}{du} = 0$$

$\frac{d^2y}{du^2} - 2\frac{dy}{du} = 0$ (Persamaan differensial dengan koefisien-koefisien tetap)

Persamaan karakteristik: $t^2 - 2t = 0$

$$t(t-2) = 0$$

$$t_1 = 0, t_2 = 2$$

Jawab umum:

$$y = C_1 e^{t_1 u} + C_2 e^{t_2 u}$$

$$y = C_1 e^0 + C_2 e^{2lnx}$$

$$y = C_1 + C_2 e^{lnx^2}$$

$$y = C_1 + C_2 x^2$$

4. Carilah Jawab umum dari $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} = 0$

Penyelesaian:

$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} = 0$……………………………………………….(1)

Misalkan: $x = e^u$ (kedua ruas kita ln kan)

$\ln x = \ln e^u$

$\ln x = u$ (turunkan terhadap x)

$$\frac{1}{x} = \frac{du}{dx}$$

Maka diperoleh: $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$ (substitusikan nilai $\frac{du}{dx}$)

$\frac{dy}{dx} = \frac{dy}{du} . \frac{1}{x}$ (kedua ruas dikali x)

$x\frac{dy}{dx} = \frac{dy}{du}$ ……………………………………..(2)

Persamaan (2) didiferensialkan terhadap x.

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{du}.\frac{dy}{dx}$ (substitusikan nilai $\frac{dy}{dx}$)

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{dy}{du}(\frac{dy}{du}.\frac{1}{x})$$

$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d^2y}{du^2}.\frac{1}{x}$ (kedua ruas dikali x)

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=\frac{d^2y}{du^2}$ (substitusikan nilai $x\frac{dy}{dx}$)

$$x^2\frac{d^2y}{dx^2}+\frac{dy}{du}=\frac{d^2y}{du^2}$$

$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$ ....(3)

Persamaan (3) didiferensialkan terhadap x.

$x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}=\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right).\frac{dy}{dx}$ (substitusikan nilai $\frac{dy}{dx}$)

$x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}=\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right)(\frac{dy}{du}.\frac{1}{x})$

$x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}=\left(\frac{1}{x}\frac{d^3y}{du^3}-\frac{1}{x}\frac{d^2y}{du^2}\right)$ (kedua ruas dikali x)

$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}$ (substitusikan nilai $x^2\frac{d^2y}{dx^2}$)

$$x^3\frac{d^3y}{dx^3}+2\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right)=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}$$

$$x^3\frac{d^3y}{dx^3}+2\frac{d^2y}{du^2}-2\frac{dy}{du}=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}$$

$$x^3\frac{d^3y}{dx^3}=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}-2\frac{d^2y}{du^2}+2\frac{dy}{du}$$

$x^3\frac{d^3y}{dx^3}=\frac{d^3y}{du^3}-3\frac{d^2y}{du^2}+2\frac{dy}{du}$ ....(4)

Substitusikan persamaan (2),(3) dan (4) ke persamaan (1)

$(\frac{d^3y}{du^3}-3\frac{d^2y}{du^2}+2\frac{dy}{du})+(3\frac{d^2y}{du^2}-3\frac{dy}{du})=0$

$\frac{d^3y}{du^3}-\frac{dy}{du}=0$ (Persamaan differensial dengan Koefisien-koefisien tetap)

Persamaan karakteristik: $t^3-t=0$

$t(t^2-1)=0$

$t=0, t=1, t=-1$

Jawab umum $y=C_1e^{t_1u}+C_2e^{t_2u}+C_3e^{t_3u}$

$y=C_1+C_2e^{lnx}+C_3e^{-lnx}$

$y=C_1+C_2x+C_3\frac{1}{x}$

5. Carilah jawab umum dari $(1+x)^3\frac{d^3y}{dx^3}+(1+x)^2\frac{d^2y}{dx^2}+3(1+x)\frac{dy}{dx}-8y=0$

Penyelesaian:

$(1+x)^3\frac{d^3y}{dx^3}+(1+x)^2\frac{d^2y}{dx^2}+3(1+x)\frac{dy}{dx}-8y=0$...........(1)

Misalkan: $(1+x)=e^u$ (kedua ruas kita ln kan)

$\ln(1+\mathrm{x})=\ln \mathrm{e}^{\mathrm{u}}$

$\ln(1+x)=u$ (turunkan terhadap x)

$$\frac{1}{(1+x)}=\frac{du}{dx}$$

Maka diperoleh : $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$ (substitusikan nilai $\frac{du}{dx}$)

$\frac{dy}{dx}=\frac{dy}{du}.\frac{1}{(1+x)}$ (kedua ruas dikali $(1+x)$)

$(1+x)\frac{dy}{dx}=\frac{dy}{du}$ ....(2)

Persamaan (2) didifferensialkan terhadap x.

$(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{dy}{du}.\frac{dy}{dx}$ (subtitusikan nilai $\frac{dy}{dx}$)

$$(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{dy}{du}(\frac{dy}{du}.\frac{1}{(1+x)})$$

$(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d^2y}{du^2}.\frac{1}{(1+x)}$ (kedua ruas dikali$(1+x)$)

$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}=\frac{d^2y}{du^2}$ (subtitusikan nilai$(1+x)\frac{dy}{dx}$)

$$(1+x)^2\frac{d^2y}{dx^2}+\frac{dy}{du}=\frac{d^2y}{du^2}$$

$(1+x)^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$ ....(3)

Persamaan (3) didifferensialkan terhadap x.

$(1+x)^2\frac{d^3y}{dx^3}+2(1+x)\frac{d^2y}{dx^2}=\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right).\frac{dy}{dx}$ (substitusikan nilai $\frac{dy}{dx}$)

$$(1+x)^2\frac{d^3y}{dx^3}+2(1+x)\frac{d^2y}{dx^2}=\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right)(\frac{dy}{du}.\frac{1}{(1+x)})$$

$(1+x)^2\frac{d^3y}{dx^3}+2(1+x)\frac{d^2y}{dx^2}=(\frac{d^3y}{du^3}\frac{1}{(1+x)})-(\frac{d^2y}{du^2}\frac{1}{(1+x)})$ (dikali (1+x))

$(1+x)^3\frac{d^3y}{dx^3}+2(1+x)^2\frac{d^2y}{dx^2}=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}$ (subs. nilai $(1+x)^2\frac{d^2y}{dx^2}$)

$$(1+x)^3\frac{d^3y}{dx^3}+2\left(\frac{d^2y}{du^2}-\frac{dy}{du}\right)=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}$$

$$(1+x)^3\frac{d^3y}{dx^3}+2\frac{d^2y}{du^2}-2\frac{dy}{du}=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}$$

$$(1+x)^3\frac{d^3y}{dx^3}=\frac{d^3y}{du^3}-\frac{d^2y}{du^2}-2\frac{d^2y}{du^2}+2\frac{dy}{du}$$

$(1+x)^3\frac{d^3y}{dx^3}=\frac{d^3y}{du^3}-3\frac{d^2y}{du^2}+2\frac{dy}{du}$ ....(4)

Substitusikan persamaan (2), (3) dan (4) ke persamaan (1)

$$\left(\frac{d^3y}{du^3} - 3\frac{d^2y}{du^2} + 2\frac{dy}{du}\right) + \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + 3\left(\frac{dy}{du}\right) - 8y = 0$$

$$\frac{d^3y}{du^3} - 2\frac{d^2y}{du^2} + 4\frac{dy}{du} - 8y = 0 \qquad \text{(Pers.Differensial koefisien tetap)}$$

Persamaan karakteristik: $t^3 - 2t^2 + 4t - 8 = 0$

$$(t-2)(t^2+4) = 0$$
$$t_1 = 2\,;\, t_{2,3} = \pm\sqrt{-4}$$
$$t_2 = 2i\,;\, t_3 = -2i$$

Jawab umum: $y = C_1e^{t_1u} + C_2e^{t_2u} + C_3e^{t_3u}$

$$y = C_1e^{2u} + C_2e^{2iu} + C_3e^{-2iu}$$
$$y = C_1e^{2u} + C_2\cos 2u + C_3\sin 2u$$
$$y = C_1e^{2\ln(1+x)} + C_2\cos 2\ln(1+x) + C_3\sin 2\ln(1+x)$$
$$y = C_1e^{\ln(1+x)^2} + C_2\cos\ln(1+x)^2 + C_3\sin\ln(1+x)^2$$
$$y = C_1(1+x)^2 + C_2\cos\ln(1+x)^2 + C_3\sin\ln(1+x)^2$$

Latihan Soal!

Carilah jawab umum dari Persamaan Differensial Linear berikut!

1. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$
2. $x^3\frac{d^3y}{dx^3} - 6x\frac{dy}{dx} = 0$
3. $x^2\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} = 0$
4. $(2+x)^3\frac{d^3y}{dx^3} - (2+x)^2\frac{d^2y}{dx^2} = 0$
5. $x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} - 18y = 0$
6. $2x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 8y = 0$
7. $2(2-x)^2\frac{d^2y}{dx^2} - 4(2-x)\frac{dy}{dx} = 0$
8. $(2+x)^3\frac{d^3y}{dx^3} + 3(2+x)^2\frac{d^2y}{dx^2} - (6+3x)\frac{dy}{dx} = 0$
9. $x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} = 0$
10. $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$

Persamaan Differensial Linier tak Homogen

Bentuk Umum :

$$\mathbf{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_n y = f(x) \ldots\ldots\ldots (1)}$$

Penyelesaian:

- Menentukan jawab umum homogen (y_h) dengan $f(x) = 0$
- Menentukan jawab khusus (y_k) dimana $f(x) \neq 0$

Maka, jawab umum P.D. linier tak homogen (y_u)

$$\boldsymbol{y_u = y_h + y_k}$$

Untuk mencari jawab khusus dari P.D. linier tak homogenya dengan memisalkan:

$$f(x) \neq 0$$

Sehingga Bentuk umum berubah menjadi :

$$f(x) = e^{ax} \cos bx\,(p_0 x^m + p_1 x^{m-1} + \ldots + p_m) + e^{ax} \sin bx\,(q_0 x^m + q_1 x^{m-1} + \ldots + q_m)$$

Dimana,

$(a, b, p_0, p_1, \ldots\ p_m, q_0, q_1, \ldots q_m)$ adalah bilangan tetap dan mungkin ada di antaranya yang sama dengan nol.

Dari $(a \pm bi)$ bukan akar dari persamaan karakteristik :

$$a_0 t^n + a_1 t^{n-1} + \ldots\ldots\ldots\ldots + a_n = 0$$

Maka sebagai fungsi percobaaan :

$y = e^{ax} \cos bx\,(k_0 x^m + k_1 x^{m-1} + \ldots + k_m) + e^{ax} \sin bx\,(l_0 x^m + l_1 x^{m-1} + \ldots + l_m)$ ....(2)

Setelah didifferensiir n kali, masing-masingnya dimasukkan ke dalam persamaan (1), maka dengan identitet harga dari $k_0, k_1, \ldots, k_m, l_0, l_1, \ldots, l_m$ dapat dicari.

Bila $(a \pm bi)$ akar lipat h dari persamaan karakteristik, maka fungsi percobaan (2) dikalikan dengan x^h.

Catatan :

Bila $a = 0, b = 0$, maka fungsi percobaan menjadi

$$y = k_0 x^m + k_1 x^{m-1} + \ldots + k_m$$

atau $(0 \pm 0i)$ adalah bukan akar

Bila $(0 \pm 0i)$ adalah akar dari persamaan karakteristik lipat h maka fungsi percobaan

$$y = (k_0x^m + k_1x^{m-1} + \ldots + k_m)x^h$$

Contoh Soal :

Untuk contoh soal 1-5, tentukanlah jawab umum persamaan differensial linier tak homogen yang diberikan.

1. $y'' - 4y = 16x^2$

Penyelesaian:

$y'' - 4y = 16x^2$ ....(1)
$y'' - 4y = 0$ (P. D. linier homogen)
Persamaan Karakteristik : $t^2 - 4 = 0$

$$t_{1,2} = \pm\sqrt{4}$$
$$t_1 = 2 \quad t_2 = -2$$

$t_1 \neq t_2$
Jawab umum homogen : $y_h = C_1e^{2x} + C_2e^{-2x}$
$y'' - 4y = 16x^2$; $f(x) = 16x^2$
$a = 0$
$b = 0$
$m = 2$
$(a \pm bi) = (0 \pm 0i) = 0$ (bukan akar dari persamaan karakteristik)
Fungsi percobaan :
$y \;= k_0x^2 + k_1x + k_2$ ......(2)
$y' = 2k_0x + k_1$ ....(3)
$y'' = 2k_0$.................................(4)
Substitusi persamaan (2) dan (4) ke persamaan (1)
$2k_0 - 4(k_0x^2 + k_1x + k_2) = 16x^2$
$2k_0 - 4k_0x^2 - 4k_1x - 4k_2 = 16x^2$
$-4k_0x^2 - 4k_1x + 2k_0 - 4k_2 = 16x^2$

- $-4k_0x^2 = 16x^2$
  $-4k_0 = 16$
  $k_0 = -4$
- $-4k_1x = 0$
  $-4k_1 = 0$

$k_1 = 0$

- $$\begin{aligned} 2k_0 - 4k_2 &= 0 \\ 2(-4) - 4k_2 &= 0 \\ -8 - 4k_2 &= 0 \\ -4k_2 &= 8 \\ k_2 &= -2 \end{aligned}$$

Substitusi nilai $k_0 = -4, k_1 = 0$, dan $k_2 = -2$ ke persamaan (2)
Jawab khusus : $y_k = -4x^2 - 2$

Jawab umum P. D. linier tak homogen : $y_u = y_h + y_k$
$$y_u = C_1 e^{2x} + C_2 e^{-2x} - 4x^2 - 2$$

2. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = e^{5x}$

Penyelesaian :

$y'' - 2y' - 3y = e^{5x}$ (1)
$y'' - 2y' - 3y = 0$ (P. D. linier homogen)
Persamaan Karakteristik : $t^2 - 2t - 3 = 0$
$$(t-3)(t+1) = 0$$
$$t_1 = 3 \qquad t_2 = -1$$
$t_1 \neq t_2$
Jawab umum homogen : $y_h = C_1 e^{3x} + C_2 e^{-x}$
$y'' - 2y' - 3y = e^{5x}$; $f(x) = e^{5x}$
$a = 5$
$b = 0$
$m = 0$
$(a \pm bi) = (5 \pm 0i) = 5$ (bukan akar dari persamaan karakteristik)
Fungsi percobaan :
$y = k_0 e^{5x}$..................(2)
$y' = 5k_0 e^{5x}$ ....(3)
$y'' = 25k_0 e^{5x}$ ....(4)
Substitusi persamaan (2), (3), dan (4) ke persamaan (1)
$$\begin{aligned} 25k_0e^{5x} - 2(5k_0e^{5x}) - 3(k_0e^{5x}) &= e^{5x} \\ 25k_0e^{5x} - 10k_0e^{5x} - 3k_0e^{5x} &= e^{5x} \\ 12k_0e^{5x} &= e^{5x} \\ 12\,k_0 &= 1 \\ k_0 &= \frac{1}{12} \end{aligned}$$

Substitusi nilai $k_0 = \frac{1}{12}$ ke persamaan (2)

Jawab khusus : $y_k = \frac{1}{12} e^{5x}$

Jawab umum P. D. linier tak homogen : $y_u = y_h + y_k$

$$y_u = C_1 e^{3x} + C_2 e^{-x} + \frac{1}{12} e^{5x}$$

3. $\frac{d^2y}{dx^2} + y = \cos x$

Penyelesaian:

$y'' + y = \cos x$ .... (1)

$y'' + y = 0$ (P. D. linier homogen)

Persamaan Karakteristik: $t^2 + 1 = 0$

$t^2 = -1$

$t_{1,2} = \pm\sqrt{-1}$

$t_1 = i \quad t_2 = -i$

t_1 dan t_2 merupakan akar − akar kompleks

Jawab umum homogen : $y_h = C_1 \cos x + C_2 \sin x$

$y'' + y = \cos x$; $f(x) = \cos x$

$a = 0$

$b = 1$

$m = 0$

$(a \pm bi) = (0 \pm i) = \pm i$ (merupakan akar lipat satu dari Pers. karakteristik)

Fungsi percobaan :

$y = (\cos x\, k_0 + \sin x\, l_0)x$

$= k_0 x \cos x + l_0 x \sin x$ .... (2)

$y' = k_0 \cos x - k_0 x \sin x + l_o \sin x + l_o x \cos x$ (3)

$y'' = -k_0 \sin x - k_0 \sin x - k_0 x \cos x + l_o \cos x + l_o \cos x - l_o x \sin x$

$= -2k_0 \sin x - k_0 x \cos x + 2l_o \cos x - l_o x \sin x$.(4)

Substitusi persamaan (2) dan (4) ke persamaan (1)

$-2k_0 \sin x - k_0 x \cos x + 2l_o \cos x - l_o x \sin x + k_0 x \cos x + l_0 x \sin x$

$= \cos x$

$-2k_0 \sin x + 2l_o \cos x = \cos x$

- $-2k_0 \sin x = 0$

  $-2k_0 = 0$

  $k_0 = 0$

- $2l_o \cos x = \cos x$

  $2l_o = 1$

$$l_o = \frac{1}{2}$$

Substitusi nilai $k_0 = 0$ *dan* $l_o = \frac{1}{2}$ ke persamaan (2)

Jawab khusus : $y_k = \frac{1}{2} x \sin x$

Jawab umum P. D. linier tak homogen : $\mathrm{y_u = y_h + y_k}$

$$\mathrm{y_u = C_1 \cos x + C_2 \sin x + \frac{1}{2} x \sin x}$$

4. $\frac{d^2y}{dx^2} + y = x^2 + 2x + e^x$

Penyelesaian :

$y'' + y = x^2 + 2x + e^x$

$y'' + y = 0$ (P. D. linier homogen)

Persamaan Karakteristik : $t^2 + 1 = 0$

$t^2 = -1$

$t_{1,2} = \pm\sqrt{-1}$

$t_1 = i \;\; t_2 = -i$

t_1 dan t_2 merupakan akar − akar kompleks

Jawab umum homogen : $y_h = C_1 \cos x + C_2 \sin x$

$y'' + y = x^2 + 2x + e^x$; $f(x) = g(x) + h(x)$

$g(x) = x^2 + 2x$

$h(x) = e^x$

$y'' + y = x^2 + 2x$........(1)

$y'' + y = e^x$ ....(2)

$y'' + y = x^2 + 2x$; $g(x) = x^2 + 2x$

$a = 0$

$b = 0$

$m = 2$

$(a \pm bi) = (0 \pm 0i) = 0$(bukan akar dari persamaan karakteristik)

Fungsi percobaan 1 :

$y = k_0x^2 + k_1x + k_2$ .... (3)

$y' = 2k_0x + k_1$ .... (4)

$y'' = 2k_0$.................................... (5)

Substitusi persamaan (3) dan (5) ke persamaan (1)

$2k_0 + k_0x^2 + k_1x + k_2 = x^2 + 2x$

$k_0x^2 + k_1x + 2k_0 + k_2 = x^2 + 2x$

- $k_0x^2 = x^2$

  $k_0 = 1$

- $k_1x = 2x$
  $k_1 = 2$
- $2k_0 + k_2 = 0$
  $2(1) + k_2 = 0$
  $2 + k_2 = 0$
  $k_2 = -2$

Substitusi nilai $k_0 = 1, k_1 = 2, \text{dan } k_2 = -2$ ke persamaan (3)
Jawab khusus 1 : $y_{k1} = x^2 + 2x - 2$

$y'' + y = e^x$; $h(x) = e^x$
$a = 1$
$b = 0$
$m = 0$
$(a \pm bi) = (1 \pm 0i) = 1$(bukan akar dari persamaan karakteristik)
Fungsi percobaan 2 :
$y = k_0e^x$ ....(6)
$y' = k_0e^x$ ....(7)
$y'' = k_0e^x$ ....(8)
Substitusi persamaan (6) dan (8) ke persamaan (2)

$$k_0e^x + k_0e^x = e^x$$
$$2k_0e^x = e^x$$
$$2k_0 = 1$$
$$k_0 = \frac{1}{2}$$

Substitusi nilai $k_0 = \frac{1}{2}$ ke persamaan (6)
Jawab khusus 2 : $y_{k2} = \frac{1}{2}e^x$
Maka, Jawab khusus : $y_k = y_{k1} + y_{k2}$

$$y_k = x^2 + 2x - 2 + \tfrac{1}{2}e^x$$

Jawab umum P. D. linier tak homogen : $y_u = y_h + y_k$

$$y_u = C_1 \cos x + C_2 \sin x + x^2 + 2x - 2 + \frac{1}{2}e^x$$

5. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2.e^x$

Penyelesaian :

$y'' - 2y' + y = x^2.e^x$............(1)

$y'' - 2y' + y = 0$ (P. D. linier homogen)

Persamaan Karakteristik : $t^2 - 2t + 1 = 0$

$(t-1)(t-1) = 0$

$t_1 = 1 \quad t_2 = 1$

$t_1 = t_2$

Jawab umum homogen : $y_h = C_1 e^x + C_2 x e^x$

$y_h = e^x(C_1 + C_2 x)$

$y'' - 2y' + y = x^2.e^x$; $f(x) = x^2.e^x$

$a = 1$

$b = 0$

$m = 2$

$(a \pm bi) = (1 \pm 0i)$

$= 1$(akar lipat 2 dari P. K. sebab sama dengan t_1 dan t_2)

Fungsi percobaan :

$y \ = e^x(k_0x^2 + k_1x + k_2)x^2$

$y \ = e^x(k_0x^4 + k_1x^3 + k_2x^2)$ ....(2)

$y' \ = e^x(k_0x^4 + k_1x^3 + k_2x^2) + e^x(4k_0x^3 + 3k_1x^2 + 2k_2x)$(3)

$y'' = e^x(k_0x^4 + k_1x^3 + k_2x^2) + e^x(4k_0x^3 + 3k_1x^2 + 2k_2x)$

$+e^x(4k_0x^3 + 3k_1x^2 + 2k_2x) + e^x(12k_0x^2 + 6k_1x + 2k_2)$ ......(4)

Substitusi persamaan (2), (3), dan (4) ke persamaan (1)

$e^x(k_0x^4 + k_1x^3 + k_2x^2) + e^x(4k_0x^3 + 3k_1x^2 + 2k_2x)$

$+ e^x(4k_0x^3 + 3k_1x^2 + 2k_2x)$

$+e^x(12k_0x^2 + 6k_1x + 2k_2) - 2e^x(k_0x^4 + k_1x^3 + k_2x^2) - 2e^x(4k_0x^3$

$+ 3k_1x^2$

$+2k_2x) + e^x(k_0x^2 + k_1x + k_2)x^2 = x^2.e^x$

$e^x(12k_0x^2 + 6k_1x + 2k_2) = x^2.e^x$

$12k_0x^2e^x + 6k_1xe^x + 2k_2e^x = x^2.e^x$

- $12k_0x^2 = x^2$

  $k_0 = \frac{1}{12}$
- $6k_1 = 0$

  $k_1 = 0$
- $2k_2 = 0$

  $k_2 = 0$

Substitusi nilai k_0, k_1, k_2ke pers (2)

$y = e^x(\frac{1}{12})\, x^4$

$y = \frac{1}{12} e^x x^4$

Maka : $y_u = y_{h+}y_k$

$$y_u = c_1 e^x + c_2 x\, e^x + \frac{1}{12} e^x x^4$$

Latihan Soal

Tentukanlah jawab umum dari Persamaan Diferensial Tak Homogen berikut:

1. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 4\sin x$
2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos 4x$
3. $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 3x + 2$
4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{2x} + 3\sin x$
5. $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 16y = e^{4x}.\sin 2x$
6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$
7. $\frac{d^4y}{dx^4} + \frac{dy}{dx} + y = \cos 2x + 3$
8. $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = e^x.\sin x$
9. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 8$
10. $\frac{d^2y}{dx^2}$ -7 $\frac{dy}{dx}$ +10 y = e^{5x} (x – 2)

Metode Variasi Parameter

BentukUmum :

$$y'' + py' + qy = r(x) \qquad \text{dimana } r(x) \neq 0$$

Solusiumum : $y_u = y_h + y_p$

Langkah 1. MenentukanPersamaanDiferensial LinearHomogen

- r(x) = 0

  $y'' + py' + qy = 0$
- Akar – akar persamaan karakteristik
  1) Jika$t_1 = t_2$ maka $y_h = c_1 e^{t_1 x} + c_2 x e^{t_2 x}$
  2) Jika$t_1 \neq t_2$ maka $y_h = c_1 e^{t_1 x} + c_2 e^{t_2 x}$
  3) Jikat = a $\pm$ bi maka $y_h = (c_1 \cos bx + c_2 \sin bx).e^{ax}$
- Solusi PersamaanDiferensial Linear homogen

$$\boldsymbol{y_h = c_1 y_1 + c_2 y_2 (\textbf{dimana } c_1 \; dan \; c_2 adalah \; konstanta)}$$

Langkah 2. Menentukany_p dengan metodeVariasi Parameter.

Solusi khusus :

Pandang $c_1 = u(x)$dan $c_2 = v(x)$

$\boldsymbol{y_p = uy_1 + vy_2}$(dimana u(x) = u dan v(x) = v)

$$y = uy_1 + vy_2 \qquad (1)$$

$$y' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

karena$u'y_1 + v'y_2 = 0$ $\quad (a)$

maka $y' = uy_1' + vy_2'$ $\quad (2)$

$$y'' = u'y_1' + uy_1'' + v'y_2' + vy_2'' \quad (3)$$

Subtitusi persamaan 1,2dan 3 ke Bentuk Umum

$y'' + py' + qy = r(x)$

$u'y_1' + uy_1'' + v'y_2' + vy_2'' + p(uy_1' + vy_2') + q(uy_1 + vy_2) = r(x)$

$u'y_1' + uy_1'' + v'y_2' + vy_2'' + upy_1' - vpy_2' + uqy_1 + vqy_2 = r(x)$

$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = r(x)$

$karena: y_1'' + py_1' + qy_1 = 0\ dan\ \ y_2'' + py_2' + qy_2 = 0$

Maka :$u'y_1' + v'y_2' = r(x)$ $\quad (b)$

Persamaan (a) dan (b) merupakan sistem persamaan linear 2 variabel

$u'y_1 + v'y_2 = 0$

$u'y_1' + v'y_2' = r(x)$

Menentukan nilai u' dan v' dengan aturan *cramer*

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 r(x)}{y_1y_2' - y_1'y_2} \qquad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 r(x)}{y_1y_2' - y_1'y_2}$$

Keterangan: W (Wronskian) = $y_1y_2' - y_1'y_2$

$$\frac{du}{dx} = -\frac{y_2 r(x)}{w} \qquad \frac{dv}{dx} = \frac{y_1 r(x)}{w}$$

$$\int du = -\int \frac{y_2 r(x)}{w}dx \qquad \int dv = \int \frac{y_1 r(x)}{w}dx$$

$$\boldsymbol{u = -\int \frac{y_2 r(x)}{w}dx} \qquad \boldsymbol{v = \int \frac{y_1 r(x)}{w}dx}$$

Langkah 3: menentukan solusi jawab

$$y = y_h + y_p$$

$$\boldsymbol{y = c_1y_1 + c_2y_2 + uy_1 + vy_2}$$

Contoh Soal:

Carilah jawab umum dari Persamaan Differensial dengan menggunakan metode variasi paramter berikut!

1. $y'' - 2y' + y = \frac{e^x}{1+x^2}$

Penyelesaian :

$y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$ ……………………………………(1)

$pk: t^2 - 2t + 1 = 0$ …………………………………………….$(PD.Homogen)$

$(t - 1)^2 = 0$

$t_1 = t_2 = 1$
$y_h = c_1 e^x + c_2 x e^x$
$untuk\ y_p:\ mis: c_1 = u(x), c_2 = v(x)$
$y_p = ue^x + vxe^x$
Maka $y_1 = e^x$ $\quad y_1' = e^x$
$y_2 = xe^x$ $\quad y_2' = e^x + xe^x$
$w = y_1 y_2' - y_1' y_2$
$w = e^x(e^x + xe^x) - e^x(xe^x)$
$w = e^{2x} + xe^{2x} - xe^{2x}$
$w = e^{2x}$
Sehingga diperoleh :

$$u = -\int \frac{y_2 r(x)}{w} dx = -\int \frac{xe^x(e^x/1 + x^2)}{e^{2x}} dx$$

$$u = -\int \frac{xe^{2x}/1 + x^2}{e^{2x}} dx$$

$$u = -\int \frac{x}{1 + x^2} dx \rightarrow mis : z = 1 + x^2$$

$$dz = 2x\, dx$$

$$dx = \frac{1}{2x}\, dz$$

$$u = -\int \frac{x}{z} . \frac{1}{2x}\, dz$$

$$u = -\frac{1}{2}\int \frac{1}{z}\, dz$$

$$u = -\frac{1}{2}\ln z$$

$$u = -\frac{1}{2}\ln(1 + x^2)$$

$$v = \int \frac{y_1 r(x)}{w} dx = \int \frac{e^x(e^x/1 + x^2)}{e^{2x}} dx$$

$$v = \int \frac{e^{2x}/1 + x^2}{e^{2x}} dx$$

$$v = \int \frac{1}{1 + x^2} dx$$

$$v = \tan^{-1} x$$

Maka

$$y_p = uy_1 + vy_2$$

$$y_p = \left[-\frac{1}{2}\ln(1 + x^2)\right] e^x + [\tan^{-1} x] xe^x$$

$$y_p = -\frac{1}{2}e^x \ln(1 + x^2) + xe^x \tan^{-1} x$$

Jadi solusi jawab dari persamaan diatas yaitu:

$$y = y_h + y_p$$

$$y = c_1e^x + c_2xe^x - \frac{1}{2}e^x \ln(1 + x^2) + xe^x \tan^{-1} x$$

2. $y'' + y = 2e^{3x}$

Penyelesaian :

$y'' + y = 2e^{3x}$

$t^2 + 1 = 0$

$t^2 = -1$

$t_{1,2} = \pm i$

$y_h = c_1 \cos x + c_2 \sin x$

$y_p = uy_1 + vy_2 = u \cos x + v \sin x$

$y_1 = \cos x \qquad y_1' = -\sin x$

$y_2 = \sin x \qquad y_2' = \cos x$

$w = \cos x \,.\, \cos x - (\sin x \,.\, (-\sin x))$

$w = cos^2x + \ sin^2x$

$w = 1$

$$u = -\int \frac{\sin x \,.\ 2e^{3x}}{1} dx$$

$$u = -2 \int \sin x \,.\ e^{3x}\, dx$$

$mis : u = \sin x \qquad dv = e^{3x}dx$

$du = \cos x\, dx \qquad v = \frac{1}{3}e^{3x}$

❖ $\int \sin x \,.\ e^{3x}dx = \sin x \,.\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} \,.\cos x\, dx$

$$= \frac{1}{3}e^{3x} \,.\sin x - \frac{1}{3}\int e^{3x} \,.\cos x\, dx$$

$$\boldsymbol{u} \rightarrow -2\int \sin x \,.\ e^{3x}\, dx = -2\left[\frac{1}{3}e^{3x} \,.\sin x - \frac{1}{3}\int e^{3x} \,.\cos x\, dx\right]$$

$$= -\frac{2}{3}e^{3x} \sin x + \frac{2}{3}\int \boldsymbol{e^{3x} \,.\cos x\, dx}$$

$misalkan : u = \cos x$

$du = -\sin x\, dx$

$dv = e^{3x}dx$

$$v = \frac{1}{3}e^{3x}$$

$$\int \boldsymbol{e^{3x}} \, . \, \mathbf{cos}\, \boldsymbol{x}\, \boldsymbol{dx} = \cos x \, . \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} . \; -\sin x \, dx$$

$$= \frac{1}{3}e^{3x}\cos x + \frac{1}{3}\int e^{3x}\sin x \, dx$$

$$= -\frac{2}{3}e^{3x}\sin x + \frac{2}{3}\left[\frac{1}{3}e^{3x}\cos x + \frac{1}{3}\int e^{3x}\sin x \, dx\right]$$

$$= -\frac{2}{3}e^{3x}\sin x + \frac{2}{9}e^{3x}\cos x + \frac{2}{9}\int e^{3x}\sin x \, dx$$

$$\boldsymbol{u} \to -2\int \sin x \, . \; e^{3x}\, dx = -\frac{2}{3}e^{3x}\sin x + \frac{2}{9}e^{3x}\cos x + \frac{2}{9}\int e^{3x}\sin x \, dx$$

$$\left(-2 - \frac{2}{9}\right)\int \sin x \, . \; e^{3x}\, dx = -\frac{2}{3}e^{3x}\sin x + \frac{2}{9}e^{3x}\cos x$$

$$\int \sin x \, . \; e^{3x}\, dx = \frac{-\frac{2}{3}e^{3x}\sin x + \frac{2}{9}e^{3x}\cos x}{-20/9}$$

$$= \frac{3}{10}e^{3x}\sin x - \frac{1}{10}e^{3x}\cos x$$

$$\boldsymbol{u} \to -2\int \sin x \, . \; e^{3x}\, dx = -2\left[\frac{3}{10}e^{3x}\sin x - \frac{1}{10}e^{3x}\cos x\right]$$

$$u = -\frac{3}{5}e^{3x}\sin x + \frac{1}{5}e^{3x}\cos x$$

$$v = \int \frac{\cos x \, . \; 2e^{3x}}{1}dx$$

$$v = 2\int \cos x \, . \; e^{3x}dx$$

$$= 2\left[\frac{1}{3}e^{3x}\cos x + \frac{1}{3}\int e^{3x}\sin x \, dx\right]$$

$$= \frac{2}{3}e^{3x}\cos x + \frac{2}{3}\int e^{3x}\sin x \, dx$$

$$= \frac{2}{3}e^{3x}\cos x + \frac{2}{3}\left[\frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x \, dx\right]$$

$$= \frac{2}{3}e^{3x}\cos x + \frac{2}{9}e^{3x}\sin x - \frac{2}{9}\int e^{3x}\cos x \, dx$$

$$2\int \cos x \, . \; e^{3x}dx \; = \frac{2}{3}e^{3x}\cos x + \frac{2}{9}e^{3x}\sin x - \frac{2}{9}\int e^{3x}\cos x \, dx$$

$$\left(2 + \frac{2}{9}\right)\int \cos x \, . \; e^{3x}dx = \frac{2}{3}e^{3x}\cos x + \frac{2}{9}e^{3x}\sin x$$

$$\int \cos x \, . \; e^{3x}dx = \frac{\frac{2}{3}e^{3x}\cos x + \frac{2}{9}e^{3x}\sin x}{20/9}$$

$$\int \cos x \,.\; e^{3x} dx = \frac{3}{10} e^{3x} \cos x + \frac{1}{10} e^{3x} \sin x$$
$$2\int \cos x \,.\; e^{3x} dx = 2\left(\frac{3}{10} e^{3x} \cos x + \frac{1}{10} e^{3x} \sin x\right)$$
$$v = \frac{3}{5} e^{3x} \cos x + \frac{1}{5} e^{3x} \sin x$$
$$y_p = uy_1 + vy_2$$
$$y_p = \left(-\frac{3}{5} e^{3x} \sin x + \frac{1}{5} e^{3x} \cos x\right) \cos x + \left(\frac{3}{5} e^{3x} \cos x + \frac{1}{5} e^{3x} \sin x\right) \sin x$$
$$y_p = -\frac{3}{5} e^{3x} \sin x \cos x + \frac{1}{5} e^{3x} cos^2 x + \frac{3}{5} e^{3x} \sin x \cos x + \frac{1}{5} e^{3x} sin^2 x$$
$$y_p = \frac{1}{5} e^{3x} cos^2 x + \frac{1}{5} e^{3x} sin^2 x$$
$$y_p = \frac{1}{5} e^{3x} (cos^2 x + sin^2 x)$$
$$y_p = \frac{1}{5} e^{3x}$$
$$y_u = y_h + y_p$$
$$y_u = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{3x}$$

3. $y'' + y = \sec x$

Penyelesaian:

$$y'' + y = \sec x$$
$$y'' + y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (PD\ Homogen)$$
$$P.K \rightarrow\ t^2 + 1 = 0$$
$$t_{1,2} = \pm i$$
$$t_1 = i\ ; t_2 = -i$$
$$y_h = c_1 \cos x + c_2 \sin x$$
$$untuk\ y_p:\ mis: c_1 = u(x), c_2 = v(x)$$

$$sehingga : y_p = u \cos x + v \sin x$$

Maka $y_1 = \cos x$ $\quad$ $y_1' = -\sin x$

$y_2 = \sin x$ $\quad$ $y_2' = \cos x$

$$w = \cos x . \cos x - (-\sin x). \sin x$$
$$w = cos^2 x + sin^2 x$$
$$w = 1$$

Sehingga diperoleh

$$u = -\int \frac{\sin x . \sec x}{1} dx \qquad v = \int \frac{\cos x \ . \ \sec x}{1} dx$$

$$u = -\int \sin x \ . \frac{1}{\cos x} dx \qquad v = \int \cos x \ . \frac{1}{\cos x} dx$$

$$u = -\int \frac{\sin x}{\cos x} dx \qquad v = \int \frac{\cos x}{\cos x} dx$$

$$u = -\int \tan x \ dx \qquad v = \int dx$$

$u = \ln|\cos x| \qquad v = x$

Maka

$y_p = uy_1 + vy_2$

$y_p = (\ln \cos x) . \cos x + x \ . \sin x$

$y_p = \cos x . \ln \cos x + x \sin x$

Jadi solusi umum dari persamaan diatas

$y = y_h + y_p$

$y = c_1 \cos x + c_2 \sin x + \cos x . \ln \cos x + x \sin x$

$$\boldsymbol{y = \cos x(c_1 + \ln \cos x) + \sin x \ (c_2 + x)}$$

4. $y'' - 2y' + 2y = e^x \sin x$

Penyelesaian:

$y'' - 2y' + 2y = e^x \sin x$

$y'' - 2y' + 2y = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (PD \ homogen)$

$P.K \rightarrow \ t^2 - 2t + 2 = 0$

$$t = \frac{2 \pm \sqrt{4-8}}{2}$$

$$t = \frac{2 \pm \sqrt{-4}}{2}$$

$$t = \frac{2 \pm 2i}{2}$$

$t = 1 \pm i$

$t_1 = 1 + i \ ; t_2 = 1 - i$

$y_h = (c_1 \cos x + c_2 \sin x)e^x$(dimana $c_1 = u$ dan $c_2 = v$)

$= c_1 e^x \cos x + c_2 e^x \sin x$

$y_p = uy_1 + vy_2 = ue^x \cos x + ve^x \sin x$

Maka$y_1 = e^x \cos x \qquad y_1' = e^x \cos x - e^x \sin x$

$y_2 = e^x \sin x \qquad y_2' = e^x \sin x + e^x \cos x$

w

$= [e^x \cos x \,.\, (e^x \sin x + e^x \cos x)]$

$- [(e^x \cos x - e^x \sin x) e^x \sin x]$

$w = e^{2x} \sin x \cos x + e^{2x} cos^2 x - e^{2x} \sin x \cos x + e^{2x} sin^2 x$

$w = e^{2x}(sin^2 x + cos^2 x)$

$w = e^{2x}$

Sehingga diperoleh

$$u = -\int \frac{e^x \sin x \,.\, e^x \sin x}{e^{2x}} dx$$

$$u = -\int sin^2 x \,.\; e^{2x} \,.\, e^{-2x}\, dx$$

$$u = -\int sin^2 x\, dx$$

$$u = \int (\frac{1}{2} \cos 2x - \frac{1}{2}) dx$$

$$u = \int -\frac{1}{2} dx + \int \frac{1}{2} \cos 2x\, dx$$

$$u = -\frac{1}{2} x + \frac{1}{2} \int \cos 2x\, dx$$

$$u = -\frac{1}{2} x + \frac{1}{2} \,.\, \frac{1}{2} \sin 2x$$

$$u = -\frac{1}{2} x + \frac{1}{4} \sin 2x$$

$$v = \int \frac{e^x \cos x \,.\, e^x \sin x}{e^{2x}} dx$$

$$v = \int \sin x \cos x\; e^{2x} . e^{-2x} dx$$

$$v = \int \sin x \cos x\, dx$$

$mis : u = \cos x$

$du = -\sin x\, dx$

$$dx = \frac{du}{-\sin x}$$

$$v = \int \sin x \,.\, u \,.\, \frac{du}{-\sin x}$$

$$v = -\int u\, du$$

$$v = -\frac{1}{2} u^2$$

$mis : v = \sin x$

$dv = \cos x\, dx$

$$dx = \frac{dv}{\cos x}$$

$$v = \int v \,.\, \cos x \,.\; \frac{dv}{\cos x}$$

$$v = \int v\, dv$$

$$v = \frac{1}{2} v^2$$

$$v = -\frac{1}{2}cos^2x \qquad\qquad v = \frac{1}{2}sin^2x$$

Maka

- $y_{p1} = uy_1 + vy_2$

$$y_{p1} = \left(-\frac{1}{2}x + \frac{1}{4}\sin 2x\right)e^x\cos x + \left(-\frac{1}{2}cos^2x\right)e^x\sin x$$

$$y_{p1} = -\frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x - \frac{1}{2}e^x cos^2x\sin x$$

- $y_{u1} = y_h + y_p$

$$y_{u1} = c_1e^x\cos x + c_2e^x\sin x - \frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x - \frac{1}{2}e^x cos^2x\sin x$$

$$y_{u1} = c_1e^x\cos x + c_2e^x\sin x - \frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x + \left(-\frac{1}{2}\sin x + \frac{1}{2}sin^3x\right)e^x$$

- $y_{p2} = uy_1 + vy_2$

$$y_{p2} = \left(-\frac{1}{2}x + \frac{1}{4}\sin 2x\right)e^x\cos x + \left(\frac{1}{2}sin^2x\right)e^x\sin x$$

$$y_{p2} = -\frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x + \frac{1}{2}e^x sin^3x$$

- $y_{u2} = y_h + y_p$

$$y_{u2} = c_1e^x\cos x + c_2e^x\sin x - \frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x + \frac{1}{2}e^x sin^3x$$

$$y_{u2} = c_1e^x\cos x + c_2e^x\sin x - \frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x + \frac{1}{2}\sin x - \frac{1}{2}cos^2x\sin x$$

Jadi solusi umum dari persamaan y_{u1} dan y_{u2}

$$y_{u1} = c_1e^x\cos x + c_2e^x\sin x - \frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x\left(-\frac{1}{2}\sin x + \frac{1}{2}sin^3x\right)e^x$$

$$y_{u2} = c_1e^x\cos x + c_2e^x\sin x - \frac{1}{2}xe^x\cos x + \frac{1}{4}e^x\sin 2x\cos x + (\frac{1}{2}\sin x - \frac{1}{2}cos^2x\sin x)e^x$$

misalkan :

$c_2 - \frac{1}{2} = a \qquad dan \qquad c_2 + \frac{1}{2} = a$

maka:

$$\boldsymbol{y_u = c_1 e^x \cos x + a\, e^x \sin x - \frac{1}{2} x e^x \cos x + \frac{1}{4} e^x \sin 2x \cos x}$$

5. $y'' - 4y = 16x$

Penyelesaian:

$y'' - 4y = 16x$

$y'' - 4y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (PD\ homogen)$

$P.K \rightarrow \ t^2 - 4 = 0$

$t_{1,2} = \pm 2$

$t_1 = 2\,;\ t_2 = -2$

$$y_h = c_1 e^{2x} + c_2 e^{-2x} (\text{dimana } c_1 = u \text{ dan } \mathrm{c}_2 = \mathrm{v})$$

$y_p = u y_1 + v y_2 = u e^{2x} + v e^{-2x}$

Maka:

$y_1 = e^{2x} y_1' = 2e^{2x}$

$y_2 = e^{-2x} y_2' = -2e^{-2x}$

$w = e^{2x}(-2e^{-2x}) - (e^{-2x} .2\, e^{2x})$

$w = -2 - 2$

$w = -4$

Sehingga diperoleh

$u = -\int \frac{e^{-2x}.16x^2}{-4} dx$

$u = 4 \int e^{-2x} .x^2\, dx$

$misalkan\!:\ m = x^2$

$dm = 2x\, dx$

$dn = e^{-2x} dx$

$n = -\frac{1}{2} e^{-2x}$

❖ $\int x^2 .\ e^{-2x}\, dx = \left(x^2 . -\tfrac{1}{2} e^{-2x}\right) - \left(\int -\tfrac{1}{2} e^{-2x} .\ 2x\right) dx$

$$= -\frac{1}{2} x^2 . e^{-2x} + \int x\boldsymbol{e^{-2x} dx}$$

$mis\!:\ u = x$

$$du = dx$$
$$dv = e^{-2x}dx$$
$$v = -\frac{1}{2}e^{-2x}$$

❖ $\int \boldsymbol{xe^{-2x}dx} = x\,.-\frac{1}{2}e^{-2x} - \int -\frac{1}{2}e^{-2x}dx$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\left(-\frac{1}{2}e^{-2x}\right)$$
$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$
$$= -e^{-2x}\left(\frac{1}{2}x + \frac{1}{4}\right)$$
$$= -\frac{1}{2}x^2\,.e^{-2x} - e^{-2x}\left(\frac{1}{2}x + \frac{1}{4}\right)$$
$$= -e^{-2x}\left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}\right)$$

$$u = 4\int x^2\ .e^{-2x}dx = 4\left[-e^{-2x}\left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}\right)\right]$$
$$u = -e^{-2x}(2x^2 + 2x + 1)$$

$$v = \int \frac{e^{2x}\ .\ 16x^2}{-4}dx$$
$$v = -4\int x^2e^{2x}dx$$
$$misalkan : u = x^2$$
$$du = 2x\,dx$$
$$dv = e^{2x}dx$$
$$v = \frac{1}{2}e^{2x}$$
$$\int x^2e^{2x}dx = x^2\ .\ \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\ .\ 2x\,dx$$
$$= \frac{1}{2}x^2e^{2x} - \int xe^{2x}dx$$
$$mis : u = x$$
$$du = dx$$
$$dv = e^{2x}dx$$
$$v = \frac{1}{2}e^{2x}$$
$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$
$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\,.\left(\frac{1}{2}e^{2x}\right)$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$$
$$= \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x}$$
$$= e^{2x}\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)$$

$$v = -4\int e^{\,2x}x^2dx = -4\left[e^{2x}\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)\right]$$
$$v = e^{2x}(-2x^2 + 2x - 1)$$

Maka:

$$y_p = uy_1 + vy_2$$
$$y_p = [(-2x^2 - 2x - 1)e^{-2x}]e^{2x} + [(-2x^2 + 2x - 1)e^{2x}]e^{-2x}$$
$$y_p = -2x^2 - 2x - 1 - 2x^2 + 2x - 1$$
$$y_p = -4x^2 - 2$$

Jadi solusi umum dari persamaan diatas

$$y = y_h + y_p$$
$$\boldsymbol{y = c_1e^{2x} + c_2e^{-2x} - 4x^2 - 2}$$

Latihan soal!

Selesaikan Solusi umum dari Pd dibawah ini dengan Metode Variasi Parameter !

1. $y'' + y = \frac{x}{e^{2x}}$
2. $y'' - 3y' + 2y = \frac{1}{e^x}$
3. $y'' + y = \csc x \cot x$
4. $y" - 3y' + 2y = e^{3x}$
5. $y'' +\ 2y' + y = 3e^{-x}$
6. $y'' + y = \sin x$
7. $\frac{d^2y}{dx^2} - 4y = x + e^x$
8. $y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$
9. $y'' - 2y' + y = e^x \sin x$
10. $y'' + y = \tan x$

BABA III
PEMETAAN LAPLACE

Sifat-sifat Pemetaan Laplace

Andaikan fungsi g terdefenisi untuk $0 \le t < \infty$, terbatas dan terintegralkan di dalam setiap selang terhingga $0 \le t \le b$, maka menurut defenisi

$$\int_0^\infty g(t)dt = \lim_{b \to \infty} \int_0^b g(t)\, dt$$

Kita katakan bahwa *integral takwajar* di ruas kiri konvergen atau divergen sesuai dengan ada atau tidak adanya limit di ruas kiri. Sebagai contoh $\int_0^\infty e^{-3t}\, dt$ konvergen tetapi $\int_0^\infty e^{3t}\, dt$ divergen. Jelaslah,

$$\int_0^\infty e^{-3t}\, dt = \lim_{b \to \infty} \int_0^b e^{-3t}\, dt = \lim_{b \to \infty} \left(-\frac{1}{3}\, e^{-3t}\Big|_o^b\right)$$
$$= \lim_{b \to \infty} \left(-\frac{1}{3}\, e^{-3b} + \frac{1}{3}\right) = \frac{1}{3},$$

Tetapi

$$\int_0^\infty e^{3t}\, dt = \lim_{b \to \infty} \int_0^b e^{3t}\, dt = \lim_{b \to \infty} \left(-\frac{1}{3}\, e^{3t}\Big|_o^b\right)$$
$$= \lim_{b \to \infty} \left(\frac{1}{3}\, e^{3b} + \frac{1}{3}\right) = \infty$$

Defenisi 1

Misalkan f(t) merupakan suatu fungsi dari t terdefenisi untuk t>0.Kemudian $\int_0^\infty e^{-st} f(t)dt$, jika ada dinamakan suatu fungsi dari s, dapat dikatakan $F(s)$. Oleh karena itu fungsi *F(s)* ini dapat dinamakan transformasi laplace dari f(t) dan dinotasikan oleh "$\mathscr{L}\{\boldsymbol{f(t)}\}$"**.** Jadi

$$\mathscr{L}\{\{\boldsymbol{f(t)}\}\} = \boldsymbol{F(s)} = \int_0^\infty e^{-st} f(t)dt,$$

Defenisi 2 :

Jika $\mathscr{L}\{\{f(t)\}\} = F(s)$ maka f(t) dinamakan Transformasi Laplace Invers dari F(s) dan dinotasikan dengan *f(t)*=$\mathscr{L}^{-1}\{F(s)\}$. Kemudian untuk mencari nilai $\mathscr{L}^{-1}\{F(s)\}$, maka kita harus mencari suatu fungsi dari t yang transformasi Laplacenya adalah F(s).

Contoh Soal!

1. Hitunglah pemetaan Laplace dari *f(t)* – 1.

   Penyelesaian

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st}(1)\, dt = \int_0^\infty e^{-st}\, dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-st}\, dt = \lim_{b \to \infty} -\left[\frac{e^{-st}}{s} \Big|\begin{matrix} b \\ 0 \end{matrix}\right]$$

$$= \lim_{b \to \infty} -\left[\frac{1}{s} - \frac{e^{-bs}}{s}\right]$$

Jika $s > 0$, limit di atas ujud dan kita peroleh

$$\int_0^\infty e^{-st} f(t)\, dt = \frac{1}{s},\ s > 0.$$

2. Hitung pemetaan Laplace dari $f(t) = e^{2t}$

Penyelesaian:

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st}\, e^{2t} dt = \int_0^\infty e^{-(s-2)t}\, dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-(s-2)t}\, dt = \lim_{b \to \infty} -\left[\frac{e^{-(s-2)t}}{s-2} \Big|\begin{matrix} b \\ 0 \end{matrix}\right]$$

$$= \lim_{b \to \infty} \left[\frac{1}{s-2} - \frac{e^{-(s-2)b}}{s-2}\right]$$

Limit ini ujud hanya jika $s > 2$. Karena itu,

$$\int_0^\infty e^{-st} f(t)\, dt = \frac{1}{s-2},\ s > 2.$$

Dalam contoh 1 dan 2 kita lihat bahwa pemetaan Laplace merupakan fungsi dari s. Kita ambil f sebagai *balikan pemetaan Laplace dari F.* Untuk maksud cara penulisan bagi balikan pemtetaan Laplace,kita perkenalkan lambang pengganti $\mathscr{L}[f(t)] = F(s)$. Kita tuliskan juga $f(t) = \mathscr{L}^{-1}[f(s)]$. Dari contoh 1 dan 2 kita lihat bahwa

$$\mathscr{L}^{-1}\left[\frac{1}{s}\right] = 1 \text{ dan } \mathscr{L}^{-1}\left[\frac{1}{s-2}\right] = e^{2t}.$$

Kita catat bahwa untuk suatu f yang diberikan, F yang berkaitan ditentukan secara tunggal (bila ini ada). Akan kita tuliskan Teorema 1, mengenai balikan pemetaan Laplace, tetapi sebelumnya kita berikan defenisi berikut.

Defenisi 2

Suatu fungsi f dikatakan kontinu bagian demi bagian pada suatu selang I, jika I dapat dibagi menjadi jumlah berhingga selang-selang bagian, di dalam selang- selang bagian itu f kontinu dan mempunyai limit kiri dan kanan yang berhingga.

Defenisi 3

Suatu fungsi f dikatakan berorde eksponensial α jika t menuju takberhingga ada bilangan M, dan T sehingga

$$|f(t)| \leq Me^{\alpha t} \text{ bila } t \geq T.$$

Atau, f dikatakan berorde eksponensial α jika ada suatu α demikian sehingga

$$\lim_{t \to \infty} |f(t)| / e^{\alpha t} = \mathscr{L},$$ dimana $\mathscr{L} = 0$ dan $\mathscr{L}$ suatu bilangan positif berhingga.

Sekarang kita berikan suatu teorema yang menjamin kekonvergenanintegral (1) tanpa bukti.

Teorema 1

Jika f kontinu bagian demi bagian pada setiap selang berhingga di dalam (0, ∞), dan jika f berorde eksponensial α dan t menuju takberhingga, integral 91) konvergen untuk s > α. Selanjutnya, jika f dan g adalah fungsi-fungsi kontinu bagian demi bagian yang pemetaan Laplace-nya ujud dan memenuhi $\mathscr{L}$ [*f*(t)] = $\mathscr{L}$ [*g*(t)], *maka f = g pada titik-titik dimana f dan g kontinu. Jadi, F(s) mempunyai balikan yang kontinu, maka f adalah tunggal.*

Teorema berikut memberikan kepada kita alat-alat yang berguna untuk menghitung pemetaan Laplace. Bukti teorema pertama merupakan akibat sederhana dari definisi dan karena itu akan dihilangkan. Lambang-lambang c_1, c_2, dan k dalam teorema-teorema berikut ini menyatakan konstanta sebarang.

Teorema 2

$$\mathscr{L}[c_1 f(t) + c_2 g(t)] = c_1 F(s) + c_2 G(s)$$

Teorema 3

Jika $F(s) = \mathscr{L}[f(t)]$, maka $F(s + k) = \mathscr{L}[e^{-kt} f(t)]$.

Contoh Soal!

3. Cari (a) $\mathscr{L}[t^2]$

Penyelesain

(a) $\mathscr{L}[t^2] = \int_0^\infty t^2 e^{-st} dt = \lim_{t \to \infty} \left[\frac{t^2 e^{-st}}{-s}\right] + \frac{2}{s}\int_0^\infty te^{-st}\, dt$

$$\lim_{t \to \infty} \left[\frac{t^2 e^{-st}}{-s}\right] + \frac{2}{s} \lim_{t \to \infty} \left[\frac{te^{-st}}{s}\right] + \frac{2}{s^2}\int_0^\infty e^{-st}\, dt$$

Jika $s > 0$, maka kedua limit diatas sama dengan nol seperti dapat dibuktikan dengan menggunakan aturan 1 hospital. Jadi,

$\mathscr{L}[t^2] = \frac{2}{s^3}, s > 0$

Untuk mempermudah tranformasi laplace invers, perhatikan tabel dibawah ini (Kartono, 1994;286)

Tabel Fungsi dan transformasi Laplace Invers

No	$\boldsymbol{F(s) = \mathscr{L}\{f(t)\}}$	$\boldsymbol{f(t) = \{f(s)\}}$
1	$\frac{1}{s}$	1
2	$\frac{1}{S^n}, n = 1,2 \ldots$	$\frac{t^{n-1}}{n-1}$
3	$\frac{1}{s-a}$	e^{at}

4	$\frac{1}{(s-a)^n}, n = 1,2 \ldots$	$\frac{1}{(n-1)}!\,(t^{n-1}e^{at})$
5	$\frac{1}{(s-a)(s-b)} a \neq b$	$\frac{1}{a-b}\left(e^{at} - e^{bt}\right)$
6	$\frac{s}{(s-a)(s-b)} a \neq b$	$\frac{1}{a-b}\left(ea^{at} - be^{bt}\right)$
7	$\frac{1}{s^2 + a^2}$	$\frac{1}{a}\sin at$
8	$\frac{s}{s^2 + a^2}$	$\text{Cos}\, at$
9	$\frac{1}{s^2 + a^2}$	$\frac{1}{a}\sinh at$
10	$\frac{s^2}{s^2 + a^2}$	$\text{Cosh}\, at$
11	$\frac{1}{(s-a)^2 + b^2}$	$\frac{1}{b}e^{at}\sin bt$
12	$\frac{s-a}{(s-a)^2 + b^2}$	$e^{at}\cos bt$
13	$\frac{1}{s(s^2 - a^2)}$	$\frac{1}{a^2}(1 - \cos at)$
14	$\frac{1}{s(s^2 - a^2)}$	$\frac{1}{a^3}(at - \sin at)$
15	$\frac{1}{(s^2 + a^2)^2}$	$\frac{1}{2a^3}(\sin at - \text{at}\cos at)$
16	$\frac{s}{(s^2 + a^2)^2}$	$\frac{1}{2}(\sin at + \text{at}\cos at)$
17	$\frac{s}{(s^2 + a^2)^2}$	$\frac{1}{2}$
18	$\frac{s}{(s^2 + a^2) + (s^2 + b^2)}, (a^2 \neq b^2)$	$\frac{1}{b^2 - a^2}(\cos at - \cos bt)$

Contoh Soal!

1. $\mathscr{L}\left(e^{-2t}+4t^3+\frac{1}{2}\sin 4t-2\cosh\frac{1}{2}t\right)$ = ...

Penyelesaian :

$\mathscr{L}\left(e^{-2t}+4t^3+\frac{1}{2}\sin 4t-2\cosh\frac{1}{2}t\right)$

$= \mathscr{L}\left(e^{-2t}\right) + 4\,\mathscr{L}\left(t^3\right) + \frac{1}{2}\,\mathscr{L}\left(\sin 4t\right) - 2\,\mathscr{L}\left(\cosh\frac{1}{2}t\right)$

$= \left(\frac{1}{s+2}\right)+4\left(\frac{3!}{s^{3+1}}\right)+\frac{1}{2}\left(\frac{4}{s^2+4^2}\right)-2\left(\frac{s}{s^2-\left(\frac{1}{2}\right)^2}\right)$

$= \left(\frac{1}{s+2}\right)+4\left(\frac{6}{s^4}\right)+\frac{1}{2}\left(\frac{4}{s^2+16}\right)-2\left(\frac{s}{s^2-\left(\frac{1}{4}\right)}\right)$

$= \frac{1}{s+2}+\frac{24}{s^4}+\frac{2}{s^2+16}-\frac{2s}{s^2-\left(\frac{1}{4}\right)}$

Maka, $\mathscr{L}\left(e^{-2t}+4t^3+\frac{1}{2}\text{s}\quad 4t-2\text{c i}\quad \frac{1}{2}t\right)$ = $\frac{1}{s+2}+\frac{24}{s^4}+\frac{2}{s^2+16}-\frac{2s}{s^2-\left(\frac{1}{4}\right)}$

2. $\mathscr{L}\left(t^2e^{2t}+t\cos 2t\right)$= ....

Penyelesaian :

$\mathscr{L}\left(t^2e^{2t}+t\cos 2t\right) = \mathscr{L}\left(t^2e^{2t}\right) + \mathscr{L}\left(t\cos 2t\right)$

$= \frac{(2-1)!}{(s-2)^2} + \frac{s^2-2^2}{\left(s^2-2^2\right)^2}$

$= \frac{1}{(s-2)^2} + \frac{s^2-4}{\left(s^2-4\right)^2}$

Maka, $\mathscr{L}(t^2e^{2t}+t\cos 2t) = \frac{1}{(s-2)^2} + \frac{s^2-4}{\left(s^2-4\right)^2}$

3. $\mathscr{L}^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=$

Penyelesaian:

$$\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=\frac{A}{(s+1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$$

x $(s+1)(s-2)^3$

$$5s^2-15s-11=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)$$

$$5s^2-15s-11=A(s^2-4s+4)(s-2)\ +B(s+1)(s^2-4s+4)+C(s^2-s-2)+D(s+1)$$

$$5s^2-15s-11=A(s^3-6s+12s-8)+B(s^3-3s^2+4)+C(s^2-s-2)+D(s+1)$$

$$5s^2-15s-11=(A+B)s^3+(-6A-3B+C)s^2+(12A-C+D)s+(-8A+4B-2C+D)$$

$3A+3B=0$ ………………(1)

$-6A-3B+C=5$ ……...(2)

(+)

$-3A+C=5$ …………..(5)

$12A-C+D=-15$ …………..(3)

$-8A+4B-2C+D=-11$ ….(4)

(-)

$20A-4B+C=-4$

$20A-4(-A)+C=-4$

$24A+C=-4$ …………….(6)

Dari persamaan (5) dan (6)

$-3A+C=5$

$24A+C=-4$

(-)

$-27A=9$

$$A=\frac{9}{-27}=-\frac{1}{3}$$

$$A=-\frac{1}{3}$$

Untuk mencari nilai C

$24A+C=-4$

$$24\left(-\frac{1}{3}\right)+C=-4$$

$-8+C=-4$

$C = 4$

Untuk mencari nilai D

$12A - C + D = -15$

$12\left(-\frac{1}{3}\right) - 4 + D = -15$

$-4 - 4 + D = -15$

$D = -15 + 8$

$D = -7$

Dengan diperolehnya nilai A,C dan D maka diperoleh nilai

$B = \frac{1}{3}$

$$\mathscr{L}^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right] =$$

$$= \mathscr{L}^{-1}\left[-\frac{1}{3(s+1)}\right] + \mathscr{L}^{-1}\left[\frac{1}{3(s-2)}\right] + \mathscr{L}^{-1}\left[\frac{4}{(s-2)^2}\right] + \mathscr{L}^{-1}\left[-\frac{7}{(s-2)^3}\right]$$

$$= -\frac{1}{3}\ \mathscr{L}^{-1}\left[\frac{1}{(s+1)}\right] + \frac{1}{3}\ \mathscr{L}^{-1}\left[\frac{1}{(s-2)}\right] + 4\ \mathscr{L}^{-1}\left[\frac{1}{(s-2)^2}\right] - \frac{7}{2}\ \mathscr{L}^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4te^{2t} - \frac{7}{2}t^2e^{2t}$$

$$= -\frac{1}{3}e^{-t} + \left(\frac{1}{3} + 4t - \frac{7}{2}t^2\right)e^{2t}$$

Sehingga Diperoleh

$$\mathscr{L}^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right] = -\frac{1}{3}e^{-t} + \left(\frac{1}{3} + 4t - \frac{7}{2}t^2\right)e^{2t}$$

Latihan Soal!

Carilah penyelesaian dari pemetaan laplace dibawah ini!

1) $\mathscr{L}\left(e^{-2t}\cos 4t + te^{-3t}\sin 4t\right)$ = ...

2) $\mathscr{L}\left[\frac{d^2I}{dt^2} + 5\frac{dI}{dt} + 6I\right] = \mathscr{L}14$

Dimana I'(0)=0 ; I(0)=-1

3) $\mathscr{L}\left[\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y\right] = \mathscr{L}[\cos t3t]$

Dimana $y^1 = 1$, y(0) = -1

4. $\mathscr{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \ldots\ldots$

5. $\mathscr{L}\left(\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y\right) = \mathscr{L}(\cos 3t)$, $y(0) = y'(0) = -1$

6. $\mathscr{L}\left(\frac{d^3y}{dt^3} + 3\frac{dy}{dt} + 4y\right) = \mathscr{L}\left(e^{-2t}\right)$, $y(0) = y'(0) = y''(0) = 1$

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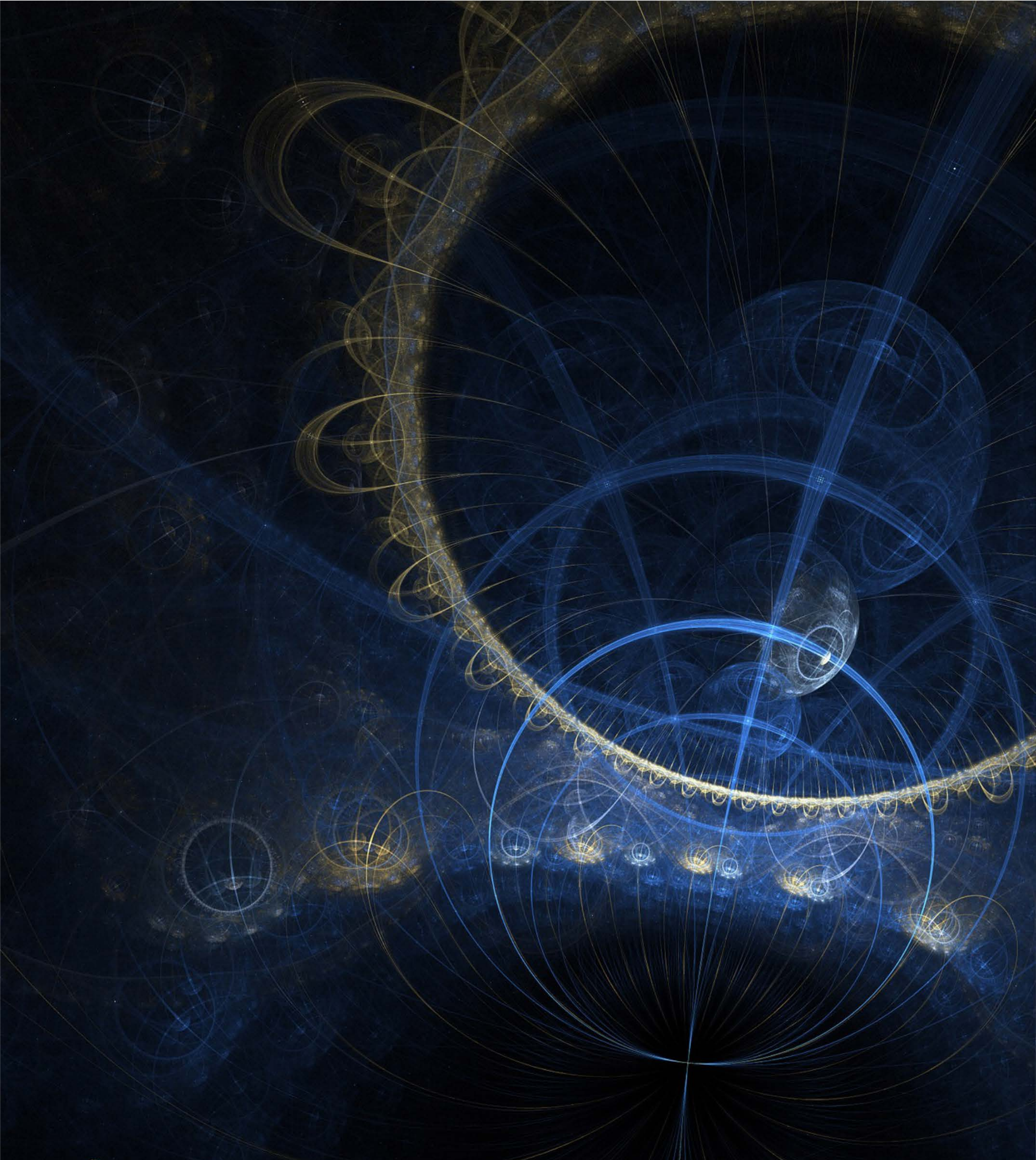

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